Numerical analysis of stochastic Poisson systems

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Outline

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IV. Splitting schemes for stochastic Poisson systems

I. Motivation

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Deterministic Hamiltonian systems (I)

Consider deterministic Hamiltonian problems of the form (Hamilton 1834):

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}(p,q), \quad \dot{q}_k = \frac{\partial H}{\partial p_k}(p,q), \quad \text{for} \quad k = 1,\ldots,d.$$

This system of differential equations describes the motion of a mechanical system with coordinates q_k and momenta p_k. Here $p = p(t) = (p_1,\ldots,p_d)^T$.

Examples: Molecular dynamics, motion of planets, mechanical systems, etc.

Remark: The Hamiltonian

$$H(p,q) = \frac{1}{2}p^T p + V(q)$$

is the total energy of the problem (kinetic energy plus potential energy).

Deterministic Hamiltonian systems (II)

Recall: Hamiltonian systems:

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}(p,q), \quad \dot{q}_k = \frac{\partial H}{\partial p_k}(p,q), \quad \text{for} \quad k = 1,\ldots,d$$

with given initial values $p(t_0) = p_{\text{init}}, q(t_0) = q_{\text{init}}$.

Property: The total energy $H(p,q)$ is an invariant:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H(p(t),q(t)) &= \frac{\partial H}{\partial p}(p(t),q(t))\,\dot{p}(t) + \frac{\partial H}{\partial q}(p(t),q(t))\,\dot{q}(t) = 0 \\ &\Rightarrow \quad H(p(t),q(t)) = \text{Constant} = H(p_{\text{init}},q_{\text{init}}) \end{aligned}$$

along the exact solution. □

Question: Design and analysis of energy-preserving numer. schemes for ODEs?

Answers 1996–: Brugnano, Celledoni, C., Gonzalez, Hairer, Iavernaro, McLachlan, McLaren, Miyatake, Owren, Quispel, Robidoux, Sato, Sun, Trigiante, Wang, Wu, Zhang, etc.

Deterministic Hamiltonian systems (III)

Recall: Hamiltonian system:

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}(p,q), \quad \dot{q}_k = \frac{\partial H}{\partial p_k}(p,q), \quad \text{for} \quad k = 1,\ldots,d.$$

with given initial values $p(t_0) = p_{\text{init}}, q(t_0) = q_{\text{init}}$.

Property: The flow $\varphi_t(p_{\text{init}}, q_{\text{init}}) := (p(t,t_0,p_{\text{init}},q_{\text{init}}), q(t,t_0,p_{\text{init}},q_{\text{init}}))$ of the above problem is symplectic (*Poincaré* 1899):

$$\varphi_t'(y)^T J \varphi_t'(y) = J \quad \text{for all} \quad y = (p,q),$$

where $J = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}$.

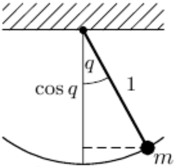
Question: Design and analysis of symplectic numerical schemes?

Answers 1956–: Bochev, de Vogelaere, Feng Kang, Hairer, Lasagni, Reich, Ruth, Sanz-Serna, Scovel, Suris, etc.

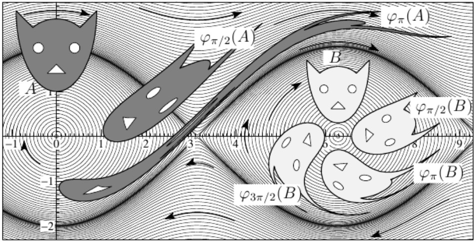
Deterministic Hamiltonian systems (IV)

The mathematical pendulum has $H(p,q) = \frac{1}{2}p^2 - \cos(q)$ and the Hamiltonian

$$\dot{p} = -\sin(q), \quad \dot{q} = p.$$



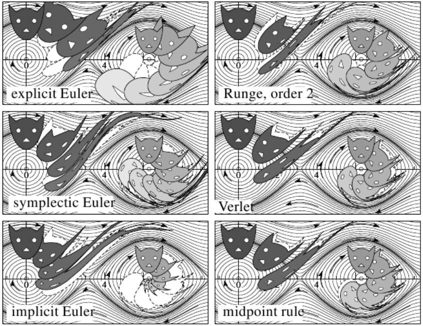
The flow is symplectic (here area preserving), phase space (q, p):



@ Book: Hairer, Wanner, Lubich, Geometric Numerical Integration 2006

Symplectic integrators

For ODE $\dot{y}(t) = f(y(t)), y(0) = y_0$ (the mathematical pendulum here):



Euler's scheme: $y_{n+1} = y_n + hf(y_n) \approx y(t_{n+1})$ is not symplectic.

The midpoint rule: $y_{n+1} = y_n + hf\left(\frac{y_n + y_{n+1}}{2}\right) \approx y(t_{n+1})$ is symplectic.

Important for long-term numerical simulations in molecular dynamics or planetary motions (movie click). Keyword: Backward Error Analysis.

@ *Book: Hairer, Wanner, Lubich, Geometric Numerical Integration* 2006

Deterministic Poisson systems

Recall: Hamiltonian systems (setting $y = (p, q)$): $\dot{y} = J^{-1}\nabla H(y)$, with the skew-symmetric constant (symplectic) matrix $J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$.

Given a Hamiltonian H and a matrix $B(y)$ (*satisfying some properties*), the ODE

$$\dot{y} = B(y)\nabla H(y)$$

is called a Poisson system. The matrix B is called the Poisson matrix.

Properties of the exact solution: The Hamiltonian is a conserved quantity. The flow of this ODE is a Poisson map (generalisation of symplecticity). One may have a Casimir function C (first integrals).

Question: Design and analysis of numerical schemes with such properties?

Answers 1988–: Channel, C., Ge, Hairer, Karasözen, McLachlan, Marsden, Reich, Scovel, etc.

Deterministic free rigid body

Recall: Poisson problem: $\dot{y} = B(y)\nabla H(y)$.

The equations for a free rigid body reads

$$\dot{y}(t) = B(y(t))\nabla H(y(t)),$$

where $y = (y_1, y_2, y_3)^\top$ represents the angular momentum in the body frame, $I = (I_1, I_2, I_3)$ are the principal moments of inertia and $B(y) = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}$.

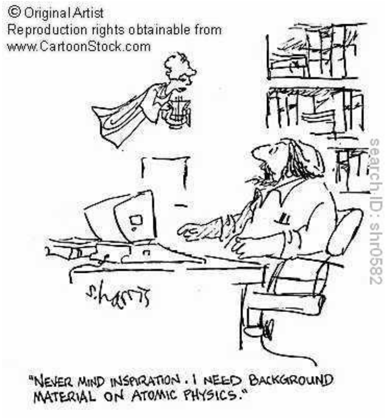
The Hamiltonian $H(y) = \frac{1}{2}\left(y_1^2/I_1 + y_2^2/I_2 + y_3^2/I_3\right)$ and the Casimir $C(y) = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2\right)$ are conserved quantities.

Further examples: Lotka–Volterra equation from population dynamics, discretisations of Euler's equations in fluid dynamics, etc.

Goal of presentation: Analyse (explicit) splitting integrators for random perturbations of Poisson systems.

A map $C(y)$ is a Casimir for the Poisson ODE $\dot{y} = B(y)\nabla H(y)$ if $\nabla C(y) B(y) = 0$ for all y. Hence $C(y)$ is also a first integral.

II. Background material on SDEs



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Stochastic differential equations (settings)

ODE. Given $f:\mathbb{R}\to\mathbb{R}$ and an initial value $x(0)$, we look for a solution to

$$\dot{x}(t) := \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t)) \Longleftrightarrow x(t) - x(0) = \int_0^t f(x(s))\,\mathrm{d}s.$$

SDE. Given $f, g:\mathbb{R}\to\mathbb{R}$ and a (non-random) initial value X_0, a stochastic process $X_t := X(t) = \{X_t(\omega)\}_{t\in[0,T]} = \{X(t,\omega)\}_{t\in[0,T]}$ is a solution to the SDE

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}W_t, \text{ with initial value } X_0$$

if X_t solves the integral equation

$$X_t - X_0 = \int_0^t f(X_s)\,\mathrm{d}s + \int_0^t g(X_s)\,\mathrm{d}W_s.$$

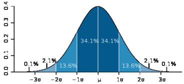
Note: X_t is a stochastic process: i. e. a random variable for each time t (on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$).

Note: Have to define W_t and the stochastic integral $\int_0^t g(X_s)\,\mathrm{d}W_s$.

Brownian motion (I)

Definition. The stochastic process W_t is a Brownian motion or standard Wiener process over $[0, T]$ if

- $W_0 = 0$ a.s.
- For any $0 \leq s < t \leq T$, $W_t - W_s \sim N(0, t - s)$ (normally dist.).

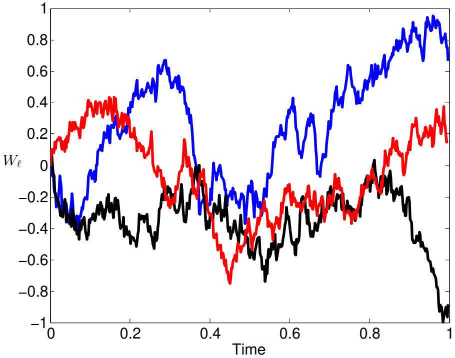


- Independent increments: For $0 \leq s \leq t \leq u \leq v \leq T$ the increments $W_t - W_s$ and $W_v - W_u$ are independent.
- W_t has a.s. cont. samples ⚠nowhere diff.

$\Longrightarrow$ At any time t, W_t is a random variable: $W_t = W_t - W_0 \sim N(0, t)$ and so $\mathbb{E}[W_t] = 0$ and $\mathbb{E}[W_t^2] = t$.

Brownian motion (II)

Numerical illustration: Discretised Brownian paths over $[0,1]$.



$\Longrightarrow$ W_t is continuous but nowhere differentiable!!

Stochastic differential equations

Recall SDE: Given $f, g\colon \mathbb{R} \to \mathbb{R}$ and a (non-random) initial value X_0, a stochastic process X_t is a solution to the SDE

$$dX_t = f(X_t)\,dt + g(X_t)\,dW_t$$

if X_t solves the integral equation

$$X_t - X_0 = \int_0^t f(X_s)\,ds + \int_0^t g(X_s)\,dW_s.$$

Note: W_s not differentiable (not even finite variation) so that we have to be careful with the definition of the above stochastic integral.

Stochastic integrals

For a deterministic function $h\colon \mathbb{R} \to \mathbb{R}$ and a partition $t_n = n\delta t$ with $\delta t = T/N$, one defines:

Deterministic Riemann integrals

$$\begin{aligned}\int_0^T h(t)\,\mathrm{d}t &= \lim_{\delta t\to 0} \sum_{n=0}^{N-1} h(t_n)(t_{n+1}-t_n)\\ &= \lim_{\delta t\to 0} \sum_{n=0}^{N-1} h\left(\frac{t_n+t_{n+1}}{2}\right)(t_{n+1}-t_n).\end{aligned}$$

Stochastic Itô integrals for stochastic process $h(t)$ (left endpoints)

$$\int_0^T h(t)\,\mathrm{d}W_t \overset{L^2}{=} \lim_{\delta t\to 0} \sum_{n=0}^{N-1} h(t_n) \overbrace{\underbrace{(W_{t_{n+1}} - W_{t_n})}_{\sim N(0,t_{n+1}-t_n)}}^{\text{random variables}}.$$

Stochastic Stratonovich integrals for stochastic process $h(t)$ (midpoint)

$$\int_0^T h(t)\circ\mathrm{d}W_t \overset{L^2}{=} \lim_{\delta t\to 0} \sum_{n=0}^{N-1} h\left(\frac{t_n+t_{n+1}}{2}\right) \overbrace{\underbrace{(W_{t_{n+1}} - W_{t_n})}_{\sim N(0,t_{n+1}-t_n)}}^{\text{random variables}}.$$

III. Drift-preserving schemes for problems with additive noise

Thanks to www.images.google.com

Stochastic Poisson problem (I)

For an integer $m > 0$ and a nice potential $V : \mathbb{R}^m \to \mathbb{R}$, consider the separable Hamiltonian

$$H(p,q) = \frac{1}{2}\sum_{j=1}^{m} p_j^2 + V(q).$$

Problem: Set $X(t) = (p(t), q(t))$ and consider Poisson system with additive noise:

$$\mathrm{d}X(t) = B(X(t))\nabla H(X(t))\,\mathrm{d}t + \begin{pmatrix}\Sigma \\ 0\end{pmatrix}\mathrm{d}W(t).$$

Here, $B(X) \in \mathbb{R}^{2m\times 2m}$ is a smooth skew-symmetric matrix, $\Sigma \in \mathbb{R}^{m\times d}$ and $W(t) \in \mathbb{R}^d$.

Examples:
Generalisation of stoch. Hamilton systems taking
$B(X) = J^{-1} = \begin{pmatrix} 0 & -Id_m \\ Id_m & 0 \end{pmatrix}$ constant matrix. Obs: odd dimension also ok!
Stochastic free rigid body ($B(X)$ not constant), Lotka–Volterra systems, etc.

Stochastic Poisson problem (II)

Recall: Poisson system with additive noise:

$$\mathrm{d}X(t) = B(X(t))\nabla H(X(t))\,\mathrm{d}t + \begin{pmatrix}\Sigma \\ 0\end{pmatrix}\mathrm{d}W(t).$$

Proposition (C., Vilmart 20*, 21): Trace formula for the energy: Along the exact solution to the above SDE, one has

$$\mathbb{E}\left[H(X(t))\right] = \mathbb{E}\left[H(X_0)\right] + \frac{1}{2}\mathrm{Tr}\left(\Sigma^\top \Sigma\right) t \quad \text{for all time} \quad t > 0.$$

The proof is done using Ito's formula.

Question: What about numerical discretisation?

Drift-preserving scheme for stochastic Poisson problem (I)

Recall: Poisson system with additive noise:

$$\mathrm{d}X(t) = B(X(t))\nabla H(X(t))\,\mathrm{d}t + \begin{pmatrix}\Sigma\\0\end{pmatrix}\mathrm{d}W(t).$$

Based on a splitting idea, we propose a new drift-preserving scheme for stochastic Poisson problem:

$$Y_1 := X_n + \begin{pmatrix}\Sigma\\0\end{pmatrix}\left(W(t_n + \frac{h}{2}) - W(t_n)\right),$$

$$Y_2 := Y_1 + hB\left(\frac{Y_1 + Y_2}{2}\right)\int_0^1 \nabla H(Y_1 + \theta(Y_2 - Y_1))\,\mathrm{d}\theta,$$

$$X_{n+1} = Y_2 + \begin{pmatrix}\Sigma\\0\end{pmatrix}\left(W(t_{n+1}) - W(t_n + \frac{h}{2})\right),$$

where $h > 0$ is the stepsize of the numerical scheme and $t_n = nh$.

Drift-preserving scheme for stochastic Poisson problem (II)

Recall: The exact solution to the Poisson system with additive noise

$$\mathrm{d}X(t) = B(X(t))\nabla H(X(t))\,\mathrm{d}t + \begin{pmatrix}\Sigma \\ 0\end{pmatrix}\mathrm{d}W(t).$$

has the trace formula for the energy

$$\mathbb{E}[H(X(t))] = \mathbb{E}[H(X_0)] + \frac{1}{2}\mathrm{Tr}\left(\Sigma^\top \Sigma\right) t \quad \text{for all time} \quad t > 0.$$

Our splitting scheme satisfies:
Theorem (C., Vilmart 20*,21): Numerical trace formula for the energy

$$\mathbb{E}[H(X_n)] = \mathbb{E}[H(X_0)] + \frac{1}{2}\mathrm{Tr}\left(\Sigma^\top \Sigma\right) t_n \quad \text{for all discrete times} \quad t_n = nh,$$

where $n \in \mathbb{N}$.

Drift-preserving scheme for stochastic Poisson problem (III)

To show: Drift-preserving scheme: $\mathbb{E}[H(X_n)] = \mathbb{E}[H(X_0)] + \frac{1}{2}\operatorname{Tr}\left(\Sigma^\top \Sigma\right) t_n$.

The first step of the drift-preserving scheme can be rewritten as

$$Y_1 = X_n + \int_{t_n}^{t_n+\frac{h}{2}} \begin{pmatrix}\Sigma\\0\end{pmatrix} \mathrm{d}W(s)$$

and an application of Itô's formula gives

$$\mathbb{E}[H(Y_1)] = \mathbb{E}[H(X_n)] + \frac{h}{4}\operatorname{Tr}\left(\Sigma^\top \Sigma\right).$$

Second step of the scheme is a deterministic energy-preserving scheme:

$$\mathbb{E}[H(Y_2)] = \mathbb{E}[H(Y_1)].$$

The last step of the numerical integrator gives

$$\begin{aligned}\mathbb{E}[H(X_{n+1})] &= \mathbb{E}[H(Y_2)] + \frac{h}{4}\operatorname{Tr}\left(\Sigma^\top \Sigma\right) = \mathbb{E}[H(Y_1)] + \frac{h}{4}\operatorname{Tr}\left(\Sigma^\top \Sigma\right)\\ &= \mathbb{E}[H(X_n)] + \frac{h}{2}\operatorname{Tr}\left(\Sigma^\top \Sigma\right).\end{aligned}$$

A recursion now completes the proof.

Drift-preservation of Casimirs

If the original ODE has a quadratic Casimir $C(X) = \frac{1}{2}X^\top AX$, with a symmetric constant matrix $A = \begin{pmatrix} a & b \\ b^\top & c \end{pmatrix}$ with $a, b, c \in \mathbb{R}^{m\times m}$, then

Theorem (C., Vilmart 20*, 21):
Trace formula for the Casimir (exact solution)

$$\mathbb{E}[C(X(t))] = \mathbb{E}[C(X_0)] + \frac{1}{2}\operatorname{Tr}\left(\Sigma^\top a\Sigma\right) t \quad \text{for all time} \quad t > 0.$$

Numerical trace formula for the Casimir (numerical solution)

$$\mathbb{E}[C(X_n)] = \mathbb{E}[C(X_0)] + \frac{1}{2}\operatorname{Tr}\left(\Sigma^\top a\Sigma\right) t_n \quad \text{for all discrete times} \quad t_n = nh,$$

where $n \in \mathbb{N}$.

A map $C(X)$ is a Casimir for the Poisson ODE $\dot{X} = B(X)\nabla H(X)$ if $\nabla C(X)B(X) = 0$ for all X. Hence $C(X)$ is also a first integral.

Rates of convergence of the drift-preserving scheme

The proposed drift-preserving scheme has the following rates of convergence under the standard setting.

Theorem (C., Vilmart 20*,21):
Mean-square order of convergence 1:

$$\left(\mathbb{E}[\|X(t_n)-X_n\|^2]\right)^{1/2} \leq Ch.$$

Weak convergence of order 2:

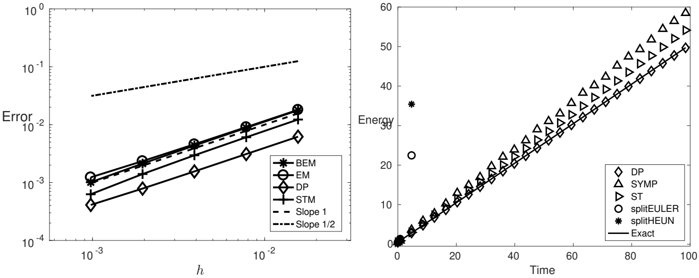
$$|\mathbb{E}[\Phi(X(t_n))]-\mathbb{E}[\Phi(X_n)]| \leq Ch^2,$$

for all test functions $\Phi \in C_P^6(\mathbb{R}^{2m},\mathbb{R})$, the space of C^6 functions with all derivatives up to order 6 with at most polynomial growth.

Linear stochastic oscillator

Problem: $\mathrm{d}X(t) = B(X(t))\nabla H(X(t))\,\mathrm{d}t + \Sigma\,\mathrm{d}W(t)$, where $X = (p, q)$, $H(p,q) = \frac{1}{2}p^2 + \frac{1}{2}q^2$ and with $\Sigma = 1$ and W scalars.
For this problem, the drift-preserving scheme is an explicit time integrator!

Mean-square error and trace formula for the energy:

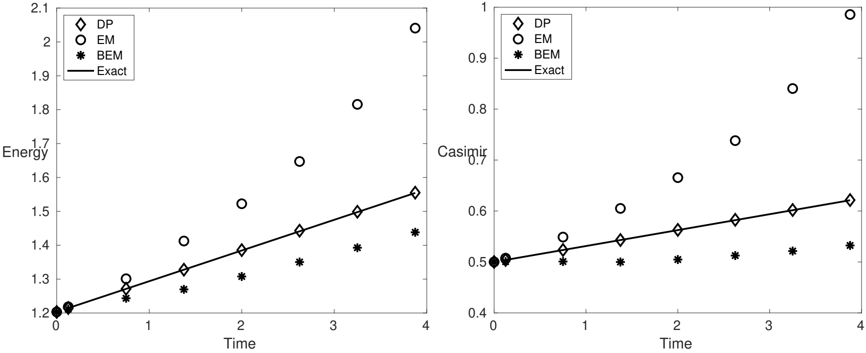


Drift-preserving scheme (DP), the splitting methods with the symplectic Euler method (SYMP), the Störmer–Verlet method (ST), the explicit Euler method (splitEULER), or the Heun method (splitHEUN).
Parameters: $(p(0), q(0)) = (0, 1)$, time interval $[0, 100]$ with 2^7 step sizes, $M_s = 10^6$ samples.

Stochastic rigid body

Problem: $\mathrm{d}X(t) = B(X(t))\nabla H(X(t))\,\mathrm{d}t + \Sigma\,\mathrm{d}W(t)$, where Hamiltonian $H(X) = \frac{1}{2}\left(X_1^2/I_1 + X_2^2/I_2 + X_3^2/I_3\right)$, and quadratic Casimir $C(X) = \frac{1}{2}\left(X_1^2 + X_2^2 + X_3^2\right)$, with Σ and W scalars (acting on first component). Here, $X = (X_1, X_2, X_3)^\top$ and moments of inertia $I = (I_1, I_2, I_3)$.

Trace formula for the energy and the Casimir:



Parameters: $X(0) = (0.8, 0.6, 0)$ and $I = (0.345, 0.653, 1)$, stepsizes $h = 4/2^5$, time interval $[0,4]$, $M_s = 2\cdot 10^6$ samples.

IV. Splitting schemes for stochastic Poisson systems

Thanks to www.images.google.com

Stochastic Lie–Poisson problems

Consider stochastic Poisson systems of the form

$$\begin{cases} \mathrm{d}y(t) = B(y(t))\nabla H(y(t))\,\mathrm{d}t + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ \mathrm{d}W_k(t), \\ y(0) = y_0, \end{cases}$$

with *Hamiltonian functions* $H, \widehat{H}_1, \dots, \widehat{H}_m \colon \mathbb{R}^d \to \mathbb{R}$, with *structure matrix* $B\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$, and with independent standard real-valued Wiener processes $W_1, \dots, W_m$.

- Skew-symmetry: for every $y \in \mathbb{R}^d$ and for all $i, j \in \{1, \dots, d\}$, one has

$$B_{ij}(y) = -B_{ji}(y).$$

- Jacobi identity: for every $y \in \mathbb{R}^d$ and for all $i, j, k \in \{1, \dots, d\}$, one has

$$\sum_{\ell=1}^{d} \left(\frac{\partial B_{ij}(y)}{\partial y_\ell} B_{\ell k}(y) + \frac{\partial B_{jk}(y)}{\partial y_\ell} B_{\ell i}(y) + \frac{\partial B_{ki}(y)}{\partial y_\ell} B_{\ell j}(y) \right) = 0.$$

- Lie–Poisson systems: B depends linearly on y.

Main results

Recall: SDE $\mathrm{d}y(t) = B(y(t))\nabla H(y(t))\,\mathrm{d}t + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ \mathrm{d}W_k(t)$.

Under technical assumptions, we:

- Prove that the flow of this SDE is a Poisson map:
  One has a.s., for all y, that $\varphi_t'(y)B(y)\varphi_t'(y)^\top = B(\varphi_t(y))$.

- Derive and analyse explicit splitting Poisson integrators for particular stochastic Lie–Poisson systems.

- Prove strong and weak convergence of such integrators, even when the coefficients of the problem are not globally Lipschitz continuous.

- Study asymptotic preserving schemes in the diffusion approximation regime.

Bréhier, C., Jahnke 2021*, 2023.

The Poisson map property *skip*

Recall: SDE $\mathrm{d}y(t) = B(y(t))\nabla H(y(t))\,\mathrm{d}t + \sum_{k=1}^m B(y(t))\nabla \widehat{H}_k(y(t)) \circ \mathrm{d}W_k(t)$.

Definition: Let D_y denote the Jacobian operator. Let $U \subset \mathbb{R}^d$ be an open set. A transformation $\varphi\colon U \to \mathbb{R}^d$ is called a *Poisson map* for the above SDE, if one has, almost surely, for all $y \in \mathbb{R}^d$,

$$D_y\varphi(y)B(y)D_y\varphi(y)^T = B(\varphi(y)).$$

Remark: Observe that a composition of Poisson maps is a Poisson map.

Theorem: Introduce the flow $(t, y) \mapsto \varphi_t(y)$ of the above SDE with coefficients of class $\mathscr{C}^3$. Assume that the flow is globally well defined and of class $\mathscr{C}^1$ with respect to the variable y. Then, for all $t \geq 0$, φ_t is a Poisson map: almost surely, for all $y \in \mathbb{R}^d$, one has

$$D_y\varphi_t(y)B(y)D_y\varphi_t(y)^T = B(\varphi_t(y)).$$

Remark: *Hong, Ruan, Sun, Wang* 21: Proof needs Darboux–Lie theorem and to rewrite SDE. Their Poisson integrators in turn need transformations and are usually implicit.

Stoch. Poisson integrators based on splitting schemes

Recall: SDE $\mathrm{d}y(t) = B(y(t))\nabla H(y(t))\,\mathrm{d}t + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ \mathrm{d}W_k(t)$.
Assumption: The Hamiltonian H can be split as follows: $H = \sum_{k=1}^{p} H_k$ for some $p \geq 1$.

Let $h > 0$ be the time step size. A numerical scheme is defined as

$$y^{[n]} = \Phi_h(y^{[n-1]}, \Delta_n W_1, \ldots, \Delta_n W_m),$$

with Wiener increments $\Delta_n W_k = W_k(nh) - W_k((n-1)h)$, $k = 1, \ldots, m$.
Provides numerical approximations: $y^{[n]} \approx y(nh)$.

Splitting schemes:

$$\Phi_h(\cdot) = \Phi_h(\cdot, \Delta W_1, \ldots, \Delta W_m) = \exp(hY_{H_p}) \circ \exp(hY_{H_{p-1}}) \circ \ldots \circ \exp(hY_{H_1})$$
$$\circ \exp(\Delta W_m Y_{\widehat{H}_m}) \circ \exp(\Delta W_{m-1} Y_{\widehat{H}_{m-1}}) \circ \ldots \circ \exp(\Delta W_1 Y_{\widehat{H}_1}),$$

where $Y_{H_k} = B\nabla H_k$, resp. $Y_{\widehat{H}_k} = B\nabla \widehat{H}_k$, denote the vector fields of the corresponding differential equations.

Convergence of the Lie–Poisson splitting schemes

Recall: SDE $\mathrm{d}y(t) = B(y(t))\nabla H(y(t))\,\mathrm{d}t + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ \mathrm{d}W_k(t)$.
Splitting scheme: $\Phi_h(\cdot) = \exp(hY_{H_p}) \circ \ldots \circ \exp(\Delta W_m Y_{\widehat{H}_m}) \circ \ldots \circ \exp(\Delta W_1 Y_{\widehat{H}_1})$.

Theorem: Assume SDE admits a Casimir function with compact level sets.
Strong convergence. Assume $B \in \mathscr{C}^2$, $H_1, \ldots, H_p \in \mathscr{C}^2$, and $\widehat{H}_1, \ldots, \widehat{H}_m \in \mathscr{C}^3$.
Then the splitting scheme has strong order of convergence equal to $1/2$: for all $T \in (0,\infty)$ and all $y_0 \in \mathbb{R}^d$, there exists a real number $c(T, y_0) \in (0,\infty)$ such that

$$\sup_{0 \le n \le N} \left(\mathbb{E}\left[\| y(nh) - y^{[n]} \|^2 \right] \right)^{1/2} \le c(T, y_0) h^{\frac{1}{2}},$$

with time step size $h = T/N$, and $y^{[0]} = y_0 = y(0)$.

Weak convergence. Assume $B \in \mathscr{C}^5$, $H_1, \ldots, H_p \in \mathscr{C}^5$, and $\widehat{H}_1, \ldots, \widehat{H}_m \in \mathscr{C}^6$.
Then the splitting scheme has weak order of convergence equal to 1: for all $T \in (0,\infty)$ and all $y_0 \in \mathbb{R}^d$, and any test function $\phi \colon \mathbb{R}^d \to \mathbb{R}$ of class $\mathscr{C}^4$ with bounded derivatives, there exists a real number $c(T, y_0, \phi) \in (0,\infty)$ such that

$$\sup_{0 \le n \le N} \left| \mathbb{E}\left[\phi\left(y(nh) \right) \right] - \mathbb{E}\left[\phi\left(y^{[n]} \right) \right] \right| \le c(T, y_0, \phi) h.$$

Main steps for the proofs *skip*

Recall: SDE $\mathrm{d}y(t) = B(y(t))\nabla H(y(t))\,\mathrm{d}t + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ \mathrm{d}W_k(t)$.
Splitting scheme: $\Phi_h(\cdot) = \exp(hY_{H_p}) \circ \ldots \circ \exp(\Delta W_m Y_{\widehat{H}_m}) \circ \ldots \circ \exp(\Delta W_1 Y_{\widehat{H}_1})$.
Strong order $1/2$, weak order 1.

1. Show a.s bounds for the exact and numerical solutions:

$$\sup_{t\in[0,T]} \|y(t)\| \le R(y_0), \quad \sup_{N\ge1} \sup_{0\le n\le N} \|y^{[n]}\| \le R(y_0),$$

   where $R(y_0) = \max_{y\in\mathbb{R}^d, C(y)=C(y_0)} \|y\|$, and $R(y_0) < \infty$.
   Use: Splitting scheme is a Poisson integrator hence preserve Casimir C.

2. Show strong and weak convergence for the auxiliary problem

$$\mathrm{d}z(t) = \sum_{k=1}^{p} f_k(z(t))\,\mathrm{d}t + \sum_{k=1}^{m} \widehat{f}_k(z(t)) \circ \mathrm{d}W_k(t),$$

   with smooth globally Lipschitz continuous functions f_k and $\widehat{f}_k$.
   Use: Fundamental theorem by Milstein and the Talay–Tubaro argument.

3. Conclude to show the convergence results for the above Poisson systems.
   Use: Combine above two steps.

Stochastic Maxwell–Bloch equations (I)

Problem: Let $d = 3$. The deterministic Maxwell–Bloch equations from laser-matter dynamics read

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = y_1 y_3 \\ \dot{y}_3 = -y_1 y_2. \end{cases}$$

This system is a deterministic Lie–Poisson system with Poisson matrix, Hamiltonian and Casimir functions given by

$$B(y) = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & 0 \\ -y_2 & 0 & 0 \end{pmatrix}, \quad H(y) = \frac{1}{2} y_1^2 + y_3, \quad C(y) = \frac{1}{2}(y_2^2 + y_3^2),$$

respectively, for all $y = (y_1, y_2, y_3) \in \mathbb{R}^3$.

Consider the following stochastic version of the Maxwell-Bloch system:

$$\mathrm{d}y = B(y)\left(\nabla H(y)\,\mathrm{d}t + \sigma_1 \nabla \widehat{H}_1(y) \circ \mathrm{d}W_1(t) + \sigma_3 \nabla \widehat{H}_3(y) \circ \mathrm{d}W_3(t)\right),$$

where $\widehat{H}_1(y) = \frac{1}{2} y_1^2$ and $\widehat{H}_3(y) = y_3$, $\sigma_1, \sigma_3 \geq 0$, driven by two independent Wiener processes W_1 and W_3.

Stochastic Maxwell–Bloch equations (II)

Recall: $\mathrm{d}y = B(y)\left(\nabla H(y)\,\mathrm{d}t + \sigma_1 \nabla \widehat{H}_1(y)\circ \mathrm{d}W_1(t) + \sigma_3 \nabla \widehat{H}_3(y)\circ \mathrm{d}W_3(t)\right)$, where $H(y) = \frac{1}{2}y_1^2 + y_3$ with $H_1(y) = \widehat{H}_1(y) = \frac{1}{2}y_1^2$ and $H_3(y) = \widehat{H}_3(y) = y_3$.

The Hamiltonian H is split as $H = H_1 + H_3$.
The two associated deterministic subsystems can be solved exactly as follows:
The deterministic subsystem corresponding with the vector field $Y_{H_1} = B\nabla H_1$ is given by

$$\begin{cases} \dot{y}_1 = 0 \\ \dot{y}_2 = y_3 y_1 \\ \dot{y}_3 = -y_2 y_1. \end{cases}$$

Observe that y_1 is constant and thus (y_2, y_3) is solution to the standard harmonic oscillator: the exact solution of the first subsystem is thus given by

$$\exp(tY_{H_1})y(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(y_1(0)t) & \sin(y_1(0)t) \\ 0 & -\sin(y_1(0)t) & \cos(y_1(0)t) \end{pmatrix} y(0)$$

for all $t \in \mathbb{R}$ and $y(0) \in \mathbb{R}^3$. *Obs.* $Y_{H_3} = B\nabla H_3$ *and stoch. parts ok!*

Stochastic Maxwell–Bloch equations (III)

Recall: $\mathrm{d}y = B(y)\big(\nabla H(y)\,\mathrm{d}t + \sigma_1 \nabla \widehat{H}_1(y) \circ \mathrm{d}W_1(t) + \sigma_3 \nabla \widehat{H}_3(y) \circ \mathrm{d}W_3(t)\big)$.

The splitting integrator then reads

$$\Phi_h = \exp(hY_{H_3}) \circ \exp(hY_{H_1}) \circ \exp(\sigma_3 \Delta W_3 Y_{\widehat{H}_3}) \circ \exp(\sigma_1 \Delta W_1 Y_{\widehat{H}_1}),$$

where for all $y \in \mathbb{R}^3$ one has

$$\exp(\sigma_1 \Delta W_1 Y_{\widehat{H}_1})y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(y_1\sigma_1\Delta W_1) & \sin(y_1\sigma_1\Delta W_1) \\ 0 & -\sin(y_1\sigma_1\Delta W_1) & \cos(y_1\sigma_1\Delta W_1) \end{pmatrix} y$$

and

$$\exp(\sigma_3 \Delta W_3 Y_{\widehat{H}_3})y = \begin{pmatrix} 1 & \sigma_3\Delta W_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} y.$$

Remark: This explicit splitting scheme is a stochastic Poisson integrator: the numerical map is a Poisson map and it preserves all Casimirs of the SDE.

Free rigid body with random inertia tensor (I)

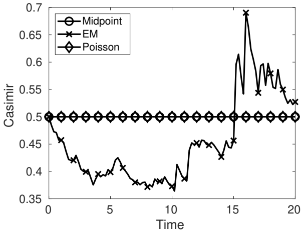
Problem: Let $H(y) = \sum_{k=1}^{3} \frac{y_k^2}{I_k}$, $\widehat{H}_k(y) = \frac{y_k^2}{\widehat{I}_k}$, for $k = 1, 2, 3$, and consider

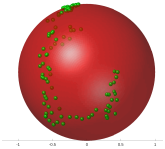
$$\mathrm{d}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = B(y)\big(\nabla H(y)\,\mathrm{d}t + \nabla \widehat{H}_1(y) \circ \mathrm{d}W_1(t) + \nabla \widehat{H}_2(y) \circ \mathrm{d}W_2(t)$$

$$+ \nabla \widehat{H}_3(y) \circ \mathrm{d}W_3(t)\big).$$

Casimir: The above SDE has a conserved quantity, the Casimir:

$$C(y) = y_1^2 + y_2^2 + y_3^2.$$



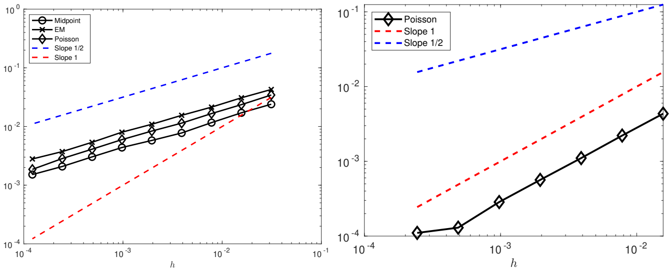


Free rigid body with random inertia tensor (II)

Problem: Let $H(y) = \sum_{n=1}^{3} \frac{y_n^2}{I_n}$, $\widehat{H}_k(y) = \frac{y_k^2}{\widehat{I}_k}$, for $k = 1, 2, 3$, and consider

$$\mathrm{d}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = B(y)\Big(\nabla H(y)\,\mathrm{d}t + \nabla \widehat{H}_1(y) \circ \mathrm{d}W_1(t) + \nabla \widehat{H}_2(y) \circ \mathrm{d}W_2(t) + \nabla \widehat{H}_3(y) \circ \mathrm{d}W_3(t)\Big).$$

Strong and weak convergence:

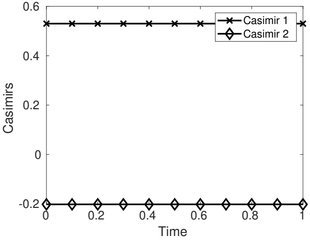


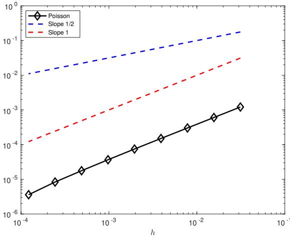
Stoch. Poisson integrators for the stochastic sine–Euler syst.

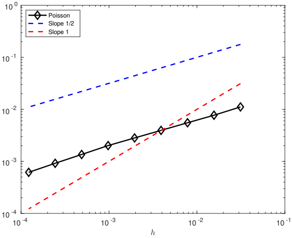
The sine–Euler equations consist of a finite-dimensional truncation of the two-dimensional Euler equations in fluid dynamics (Zeitlin 1991).

We consider random perturbations of such systems (horrible equations).

2 Casimirs (quadratic and cubic), ms order 1 (one noise), ms order 1/2 (3 noises):







Thanks for your attention!!

David Cohen, Gilles Vilmart: *Drift-preserving numerical integrators for stochastic Poisson systems*, 2020*, Int. J. Comput. Math, 2021

Charles-Edouard Bréhier, David Cohen, Tobias Jahnke: *Splitting integrators for stochastic Lie–Poisson systems*, 2021*, to appear Math. Comp 2023

Thanks to www.images.google.com and Konstantinos Dareiotis

Strang version

SDE

$$\mathrm{d}y(t) = B(y(t))\nabla H(y(t))\,\mathrm{d}t + \sum_{k=1}^{m} B(y(t))\nabla \widehat{H}_k(y(t)) \circ \mathrm{d}W(t)$$

with $H(y) = \sum_{k=1}^{p} H_k$.
Strang version

$$\Phi_h(\cdot) = \exp(hY_{H_p}) \circ \ldots \circ \exp(hY_{H_1})$$
$$\circ \exp(\Delta W/2 Y_{\widehat{H}_1}) \circ \ldots \exp(\Delta W Y_{\widehat{H}_m}) \circ \ldots \circ \exp(\Delta W/2 Y_{\widehat{H}_1})$$

Order: $1/2$ in mean-square sense and 2 in the weak sense.