Multilevel adaptivity for stochastic finite element methods

Alex Bespalov
University of Birmingham

Collaborators: **Dirk Praetorius** (TU Vienna)
Leonardo Rocchi (University of Birmingham)
Michele Ruggeri (University of Strathclyde)
David Silvester (University of Manchester)
Feng Xu (University of Manchester)

Irish Numerical Analysis Forum
26 January 2021

EPSRC
Engineering and Physical Sciences Research Council

UNIVERSITY OF BIRMINGHAM

What is this talk about...

* Computational methods for uncertainty quantification (UQ)
  - PDEs with uncertain or parameter-dependent inputs
  - forward UQ: propagation of uncertainty from the inputs/data to the output/solution
  - approximations of the input-output map
* Numerical solution of elliptic PDE problems with parametric or uncertain inputs using
  - stochastic Galerkin FEM
  - stochastic collocation FEM
* Design and analysis of adaptive algorithms
  - focus on multilevel adaptivity

Parametric model problem

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$\begin{aligned} -\nabla_x \cdot (a(x, \mathbf{y}) \nabla_x u(x, \mathbf{y})) &= f(x) \qquad & x \in D,\ \mathbf{y} \in \Gamma, \\ u(x, \mathbf{y}) &= 0 \qquad & x \in \partial D,\ \mathbf{y} \in \Gamma \end{aligned}$$

- Domains
  - $D \subset \mathbb{R}^2 \rightsquigarrow$ physical domain
  - $\Gamma := [-1, 1]^{\mathbb{N}}$ or $\Gamma := [-1, 1]^M \rightsquigarrow$ parameter domain

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) = f(x) \qquad x \in D,\ \mathbf{y} \in \Gamma,$$
$$u(x,\mathbf{y}) = 0 \qquad x \in \partial D,\ \mathbf{y} \in \Gamma$$

- Domains
  - $D \subset \mathbb{R}^2 \rightsquigarrow$ physical domain
  - $\Gamma := [-1,1]^{\mathbb{N}}$ or $\Gamma := [-1,1]^M \rightsquigarrow$ parameter domain

Remark: parameters $y_1, y_2, \ldots$ can be seen as images (observations) of independent real-valued random variables with cumulative distribution functions $\pi_1(y_1), \pi_2(y_2), \ldots$. Then, the joint cumulative distribution function is defined as

$$\pi(\mathbf{y}) := \prod_{m=1}^{\infty} \pi_m(y_m), \quad \text{and} \quad \int_{-1}^{1} d\pi_m(y_m) = \int_{\Gamma} d\pi(\mathbf{y}) = 1.$$

Parametric model problem

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$
\begin{aligned}
-\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) &= f(x) \qquad && x \in D,\ \mathbf{y} \in \Gamma, \\
u(x,\mathbf{y}) &= 0 && x \in \partial D,\ \mathbf{y} \in \Gamma
\end{aligned}
$$

- Domains
  - $D \subset \mathbb{R}^2 \rightsquigarrow$ physical domain
  - $\Gamma := [-1,1]^{\mathbb{N}}$ or $\Gamma := [-1,1]^M \rightsquigarrow$ parameter domain
- Challenges for numerics (and analysis)
  - high-dimensional parameter domain $\rightsquigarrow$ 'curse of dimensionality'
  - guaranteed and reliable error control in approximations (rigorous *a posteriori* error analysis)
  - tuning of spatial and stochastic components of approximations
  - adaptive algorithms that are provably convergent with optimal rates

Parametric model problem... and numerical solution methods

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$
\begin{aligned}
-\nabla_x \cdot (a(x, \mathbf{y}) \nabla_x u(x, \mathbf{y})) &= f(x) \qquad && x \in D, \ \mathbf{y} \in \Gamma, \\
u(x, \mathbf{y}) &= 0 \qquad && x \in \partial D, \ \mathbf{y} \in \Gamma
\end{aligned}
$$

- Monte Carlo sampling (vanilla MC, Quasi-MC, Multilevel MC)
  - effective and dimension independent
  - estimating the moments of the solution

Parametric model problem... and numerical solution methods

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x, \mathbf{y}) \nabla_x u(x, \mathbf{y})) = f(x) \qquad x \in D, \ \mathbf{y} \in \Gamma,$$
$$u(x, \mathbf{y}) = 0 \qquad x \in \partial D, \ \mathbf{y} \in \Gamma$$

- Monte Carlo sampling (vanilla MC, Quasi-MC, Multilevel MC)
  - effective and dimension independent
  - estimating the moments of the solution
- 'Surrogate' approximations that are functions of parameters, $u(x, \mathbf{y}) \approx u_\bullet(x, \mathbf{y})$

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$\begin{aligned} -\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) &= f(x) \qquad && x \in D,\ \mathbf{y} \in \Gamma, \\ u(x,\mathbf{y}) &= 0 && x \in \partial D,\ \mathbf{y} \in \Gamma \end{aligned}$$

- Monte Carlo sampling (vanilla MC, Quasi-MC, Multilevel MC)
  - effective and dimension independent
  - estimating the moments of the solution

- 'Surrogate' approximations that are functions of parameters, $u(x,\mathbf{y}) \approx u_\bullet(x,\mathbf{y})$
  - $L^2_\pi(\Gamma) = \mathrm{span}\{\boldsymbol{\Psi}_n : n \in \mathbb{N}\}$ or $\mathrm{span}\{\boldsymbol{\Psi}_n : n = 1,\ldots,N\} \subset L^2_\pi(\Gamma)$, $N \in \mathbb{N}$

Parametric model problem... and numerical solution methods

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) = f(x) \qquad x \in D,\ \mathbf{y} \in \Gamma,$$
$$u(x,\mathbf{y}) = 0 \qquad x \in \partial D,\ \mathbf{y} \in \Gamma$$

- Monte Carlo sampling (vanilla MC, Quasi-MC, Multilevel MC)
  - effective and dimension independent
  - estimating the moments of the solution

- 'Surrogate' approximations that are functions of parameters, $u(x,\mathbf{y}) \approx u_\bullet(x,\mathbf{y})$
  - $L^2_\pi(\Gamma) = \mathrm{span}\{\Psi_n : n \in \mathbb{N}\}$ or $\mathrm{span}\{\Psi_n : n = 1,\dots,N\} \subset L^2_\pi(\Gamma)$, $N \in \mathbb{N}$
  - if $u(x,\mathbf{y}) \in L^2_\pi(\Gamma; H^1_0(D))$, then

$$u(x,\mathbf{y}) \approx \sum_{n=1}^{N} \underbrace{u_n(x)}_{\in H^1_0(D)} \Psi_n(\mathbf{y})$$

Parametric model problem... and numerical solution methods

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$
\begin{aligned}
-\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) &= f(x) \qquad x \in D,\ \mathbf{y} \in \Gamma, \\
u(x,\mathbf{y}) &= 0 \qquad x \in \partial D,\ \mathbf{y} \in \Gamma
\end{aligned}
$$

- Monte Carlo sampling (vanilla MC, Quasi-MC, Multilevel MC)
  - effective and dimension independent
  - estimating the moments of the solution

- 'Surrogate' approximations that are functions of parameters, $u(x,\mathbf{y}) \approx u_\bullet(x,\mathbf{y})$
  - $L^2_\pi(\Gamma) = \mathrm{span}\{\Psi_n : n \in \mathbb{N}\}$ or $\mathrm{span}\{\Psi_n : n = 1,\dots,N\} \subset L^2_\pi(\Gamma)$, $N \in \mathbb{N}$
  - if $u(x,\mathbf{y}) \in L^2_\pi(\Gamma; H^1_0(D))$, then

$$
u(x,\mathbf{y}) \approx \sum_{n=1}^{N} \underbrace{u_n(x)}_{\in H^1_0(D)} \Psi_n(\mathbf{y}) \approx \sum_{n=1}^{N} \underbrace{u_{\bullet n}(x)}_{\text{FE sols.}} \Psi_n(\mathbf{y}) =: u_\bullet(x,\mathbf{y})
$$

Parametric model problem... and numerical solution methods

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x, \mathbf{y}) \nabla_x u(x, \mathbf{y})) = f(x) \qquad x \in D,\ \mathbf{y} \in \Gamma,$$
$$u(x, \mathbf{y}) = 0 \qquad x \in \partial D,\ \mathbf{y} \in \Gamma$$

- Monte Carlo sampling (vanilla MC, Quasi-MC, Multilevel MC)
  - effective and dimension independent
  - estimating the moments of the solution
- 'Surrogate' approximations that are functions of parameters, $u(x, \mathbf{y}) \approx u_\bullet(x, \mathbf{y})$
  - $L^2_\pi(\Gamma) = \mathrm{span}\{\Psi_n : n \in \mathbb{N}\}$ or $\mathrm{span}\{\Psi_n : n = 1, \ldots, N\} \subset L^2_\pi(\Gamma)$, $N \in \mathbb{N}$
  - if $u(x, \mathbf{y}) \in L^2_\pi(\Gamma; H^1_0(D))$, then

$$u(x, \mathbf{y}) \approx \sum_{n=1}^{N} \underbrace{u_n(x)}_{\in H^1_0(D)} \Psi_n(\mathbf{y}) \approx \sum_{n=1}^{N} \underbrace{u_{\bullet n}(x)}_{\text{FE sols.}} \Psi_n(\mathbf{y}) =: u_\bullet(x, \mathbf{y})$$

  - candidates for Ψ_n: orthogonal polynomials, Lagrange basis functions, ...

Parametric model problem: affine-parametric coefficient (1/2)

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x, \mathbf{y}) \nabla_x u(x, \mathbf{y})) = f(x) \qquad x \in D, \ \mathbf{y} \in \Gamma,$$
$$u(x, \mathbf{y}) = 0 \qquad x \in \partial D, \ \mathbf{y} \in \Gamma$$

- Affine-parametric diffusion coefficient
  - $\Gamma := [-1, 1]^{\mathbb{N}} \rightsquigarrow$ parameter domain
  - $a(x, \mathbf{y}) = a_0(x) + \sum_{m \in \mathbb{N}} y_m a_m(x) \quad$ for $x \in D$, $\mathbf{y} = (y_m)_{m \in \mathbb{N}} \in \Gamma$
  - $0 < a_0^{\min} \le a_0(x) \le a_0^{\max} < \infty \quad$ for almost all $x \in D$
  - $\tau := \frac{1}{a_0^{\min}} \left\| \sum_{m \in \mathbb{N}} |a_m| \right\|_{L^\infty(D)} < 1 \quad \& \quad \sum_{m \in \mathbb{N}} \|a_m\|_{L^\infty(D)} < \infty$

Remark: $a_0(x)$ typically represents the mean field, i.e., $a_0(x) = \int_\Gamma a(x, \mathbf{y}) d\pi(\mathbf{y})$.

Parametric model problem: affine-parametric coefficient (2/2)

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$\begin{aligned} -\nabla_x \cdot (a(x, \mathbf{y}) \nabla_x u(x, \mathbf{y})) &= f(x) \qquad x \in D, \ \mathbf{y} \in \Gamma, \\ u(x, \mathbf{y}) &= 0 \qquad x \in \partial D, \ \mathbf{y} \in \Gamma \end{aligned}$$

- $\mathbb{X} := H^1_0(D)$, $\mathbb{P} := L^2_\pi(\Gamma)$, $\pi(\mathbf{y}) = \prod_{m \in \mathbb{N}} \pi_m(y_m)$; $\quad \mathbb{V} := L^2_\pi(\Gamma; \mathbb{X}) \cong \mathbb{X} \otimes \mathbb{P}$

Parametric model problem: affine-parametric coefficient (2/2)

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) = f(x) \qquad x \in D,\ \mathbf{y} \in \Gamma,$$
$$u(x,\mathbf{y}) = 0 \qquad x \in \partial D,\ \mathbf{y} \in \Gamma$$

- $\mathbb{X} := H^1_0(D)$, $\mathbb{P} := L^2_\pi(\Gamma)$, $\pi(\mathbf{y}) = \prod_{m\in\mathbb{N}} \pi_m(y_m)$; $\quad \mathbb{V} := L^2_\pi(\Gamma;\mathbb{X}) \cong \mathbb{X} \otimes \mathbb{P}$
- Bilinear forms on $\mathbb{V}$
  - $B_0(u,v) := \int_\Gamma \int_D a_0(x)\nabla u(x,\mathbf{y}) \cdot \nabla v(x,\mathbf{y})\,\mathrm{d}x\,\mathrm{d}\pi(\mathbf{y})$
  - $B(u,v) := B_0(u,v) + \sum_{m\in\mathbb{N}} \int_\Gamma y_m \int_D a_m(x)\nabla u(x,\mathbf{y}) \cdot \nabla v(x,\mathbf{y})\,\mathrm{d}x\,\mathrm{d}\pi(\mathbf{y})$

Parametric model problem: affine-parametric coefficient (2/2)

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x, \mathbf{y})\nabla_x u(x, \mathbf{y})) = f(x) \qquad x \in D, \ \mathbf{y} \in \Gamma,$$
$$u(x, \mathbf{y}) = 0 \qquad x \in \partial D, \ \mathbf{y} \in \Gamma$$

- $\mathbb{X} := H_0^1(D)$, $\mathbb{P} := L^2_\pi(\Gamma)$, $\pi(\mathbf{y}) = \prod_{m\in\mathbb{N}} \pi_m(y_m)$; $\quad \mathbb{V} := L^2_\pi(\Gamma; \mathbb{X}) \cong \mathbb{X} \otimes \mathbb{P}$
- Bilinear forms on $\mathbb{V}$
  - $B_0(u, v) := \int_\Gamma \int_D a_0(x)\nabla u(x, \mathbf{y}) \cdot \nabla v(x, \mathbf{y})\, \mathrm{d}x\, \mathrm{d}\pi(\mathbf{y})$
  - $B(u, v) := B_0(u, v) + \sum_{m\in\mathbb{N}} \int_\Gamma y_m \int_D a_m(x)\nabla u(x, \mathbf{y}) \cdot \nabla v(x, \mathbf{y})\, \mathrm{d}x\, \mathrm{d}\pi(\mathbf{y})$

Weak formulation: given $f \in L^2(D)$, find $u \in \mathbb{V}$ such that

$$B(u, v) = F(v) := \int_\Gamma \int_D f(x)v(x, \mathbf{y})\, \mathrm{d}x\, \mathrm{d}\pi(\mathbf{y}) \quad \text{for all } v \in \mathbb{V} \qquad (\star)$$

Parametric model problem: affine-parametric coefficient (2/2)

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$
\begin{aligned}
-\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) &= f(x) \qquad && x \in D,\ \mathbf{y} \in \Gamma, \\
u(x,\mathbf{y}) &= 0 && x \in \partial D,\ \mathbf{y} \in \Gamma
\end{aligned}
$$

- $\mathbb{X} := H_0^1(D)$, $\mathbb{P} := L_\pi^2(\Gamma)$, $\pi(\mathbf{y}) = \prod_{m\in\mathbb{N}} \pi_m(y_m)$; $\quad \mathbb{V} := L_\pi^2(\Gamma;\mathbb{X}) \cong \mathbb{X} \otimes \mathbb{P}$
- Bilinear forms on $\mathbb{V}$
  - $B_0(u,v) := \int_\Gamma \int_D a_0(x)\nabla u(x,\mathbf{y}) \cdot \nabla v(x,\mathbf{y})\,\mathrm{d}x\,\mathrm{d}\pi(\mathbf{y})$
  - $B(u,v) := B_0(u,v) + \sum_{m\in\mathbb{N}} \int_\Gamma y_m \int_D a_m(x)\nabla u(x,\mathbf{y}) \cdot \nabla v(x,\mathbf{y})\,\mathrm{d}x\,\mathrm{d}\pi(\mathbf{y})$

Weak formulation: given $f \in L^2(D)$, find $u \in \mathbb{V}$ such that

$$B(u,v) = F(v) := \int_\Gamma \int_D f(x)v(x,\mathbf{y})\,\mathrm{d}x\,\mathrm{d}\pi(\mathbf{y}) \quad \text{for all } v \in \mathbb{V} \qquad (\star)$$

[Schwab, Gittelson '11]: the assumptions on $a(x,\mathbf{y})$ ensure the wellposedness of $(\star)$.

Stochastic Galerkin FEM (1/2)

- Finite dimensional subspace

$$\mathbb{V}_\bullet \subset \mathbb{V} \cong \mathbb{X} \otimes \mathbb{P}$$

Galerkin projection:

$$\text{find} \quad u_\bullet \in \mathbb{V}_\bullet \text{ such that } \; B(u_\bullet, v_\bullet) = F(v_\bullet) \quad \text{for all } v \in \mathbb{V}_\bullet$$

Stochastic Galerkin FEM (1/2)

- Finite dimensional subspace

$$\mathbb{V}_\bullet \subset \mathbb{V} \cong \mathbb{X} \otimes \mathbb{P}$$

Galerkin projection:

$$\text{find} \quad u_\bullet \in \mathbb{V}_\bullet \text{ such that } \quad B(u_\bullet, v_\bullet) = F(v_\bullet) \quad \text{for all } v \in \mathbb{V}_\bullet$$

- Galerkin orthogonality

$$B(u - u_\bullet, v_\bullet) = 0 \quad \text{for all } v_\bullet \in \mathbb{V}_\bullet$$

- Best approximation property

$$|||\, u - u_\bullet \,||| = \min_{v_\bullet \in \mathbb{V}_\bullet} |||\, u - v_\bullet \,|||, \quad \text{where} \quad |||\, \cdot \,||| := B(\cdot,\cdot)^{1/2}$$

Stochastic Galerkin FEM (1/2)

- Finite dimensional subspace

$$\mathbb{V}_\bullet \subset \mathbb{V} \cong \mathbb{X} \otimes \mathbb{P}$$

Galerkin projection:

$$\text{find} \quad u_\bullet \in \mathbb{V}_\bullet \text{ such that } \quad B(u_\bullet, v_\bullet) = F(v_\bullet) \quad \text{for all } v \in \mathbb{V}_\bullet$$

- Galerkin orthogonality

$$B(u - u_\bullet, v_\bullet) = 0 \quad \text{for all } v_\bullet \in \mathbb{V}_\bullet$$

- Best approximation property

$$|||\, u - u_\bullet \,||| = \min_{v_\bullet \in \mathbb{V}_\bullet} |||\, u - v_\bullet \,|||, \quad \text{where} \quad |||\, \cdot \,||| := B(\cdot,\cdot)^{1/2}$$

- Main question: how to choose $\mathbb{V}_\bullet$?

Stochastic Galerkin FEM (2/2)

- $\{P_\nu : \nu \in \mathfrak{I}\}$ is a countable orthonormal polynomial basis of $\mathbb{P} = L^2_\pi(\Gamma)$
- gPC expansion: $\mathbb{V} \ni u(x, \mathbf{y}) = \sum_{\nu \in \mathfrak{I}} u_\nu(x) P_\nu(\mathbf{y})$ with unique $u_\nu \in \mathbb{X} = H^1_0(D)$

Stochastic Galerkin FEM (2/2)

- $\{P_\nu : \nu \in \mathfrak{I}\}$ is a countable orthonormal polynomial basis of $\mathbb{P} = L^2_\pi(\Gamma)$
- gPC expansion: $\mathbb{V} \ni u(x, \mathbf{y}) = \sum_{\nu \in \mathfrak{I}} u_\nu(x) P_\nu(\mathbf{y})$ with unique $u_\nu \in \mathbb{X} = H^1_0(D)$
- $\mathfrak{I} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \#\mathrm{supp}(\nu) < \infty\}$ where $\mathrm{supp}(\nu) = \{m \in \mathbb{N} : \nu_m \neq 0\}$

Stochastic Galerkin FEM (2/2)

- $\{P_\nu : \nu \in \mathfrak{I}\}$ is a countable orthonormal polynomial basis of $\mathbb{P} = L^2_\pi(\Gamma)$
- gPC expansion: $\mathbb{V} \ni u(x, \mathbf{y}) = \sum_{\nu \in \mathfrak{I}} u_\nu(x) P_\nu(\mathbf{y})$ with unique $u_\nu \in \mathbb{X} = H^1_0(D)$
- $\mathfrak{I} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \#\mathrm{supp}(\nu) < \infty\}$ where $\mathrm{supp}(\nu) = \{m \in \mathbb{N} : \nu_m \neq 0\}$

Example

$\nu = (2, 5, 0, 3, 0, 0, 0 \ldots) \rightsquigarrow \mathrm{supp}(\nu) = \{1, 2, 4\}$

$\rightsquigarrow P_\nu(\mathbf{y}) = P_2(y_1)\, P_5(y_2)\, P_3(y_4)$

Stochastic Galerkin FEM (2/2)

- $\{P_\nu : \nu \in \mathfrak{I}\}$ is a countable orthonormal polynomial basis of $\mathbb{P} = L^2_\pi(\Gamma)$
- gPC expansion: $\mathbb{V} \ni u(x,\mathbf{y}) = \sum_{\nu\in\mathfrak{I}} u_\nu(x) P_\nu(\mathbf{y})$ with unique $u_\nu \in \mathbb{X} = H^1_0(D)$
- $\mathfrak{I} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \#\mathrm{supp}(\nu) < \infty\}$ where $\mathrm{supp}(\nu) = \{m \in \mathbb{N} : \nu_m \neq 0\}$
- Discretisation in the parameter domain
  - finite index set $\mathfrak{P}_\bullet \subset \mathfrak{I} \implies \mathbb{P}_\bullet = \mathrm{span}\{P_\nu : \nu \in \mathfrak{P}_\bullet\} \subset \mathbb{P} = L^2_\pi(\Gamma)$
  - semidiscrete approximation via truncation of gPC expansion

$$u(x,\mathbf{y}) \approx \sum_{\nu\in\mathfrak{P}_\bullet} u_\nu(x) P_\nu(\mathbf{y}) \in \mathbb{X} \otimes \mathbb{P}_\bullet \quad \text{with coefficients } u_\nu \in \mathbb{X} = H^1_0(D)$$

Stochastic Galerkin FEM (2/2)

- $\{P_\nu : \nu \in \mathfrak{I}\}$ is a countable orthonormal polynomial basis of $\mathbb{P} = L^2_\pi(\Gamma)$
- gPC expansion: $\mathbb{V} \ni u(x, \mathbf{y}) = \sum_{\nu \in \mathfrak{I}} u_\nu(x) P_\nu(\mathbf{y})$ with unique $u_\nu \in \mathbb{X} = H^1_0(D)$
- $\mathfrak{I} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \#\mathrm{supp}(\nu) < \infty\}$ where $\mathrm{supp}(\nu) = \{m \in \mathbb{N} : \nu_m \neq 0\}$

- Discretisation in the parameter domain
  - finite index set $\mathfrak{P}_\bullet \subset \mathfrak{I} \implies \mathbb{P}_\bullet = \mathrm{span}\{P_\nu : \nu \in \mathfrak{P}_\bullet\} \subset \mathbb{P} = L^2_\pi(\Gamma)$
  - semidiscrete approximation via truncation of gPC expansion

$$u(x, \mathbf{y}) \approx \sum_{\nu \in \mathfrak{P}_\bullet} u_\nu(x) P_\nu(\mathbf{y}) \in \mathbb{X} \otimes \mathbb{P}_\bullet \quad \text{with coefficients } u_\nu \in \mathbb{X} = H^1_0(D)$$

- Discretisations in the physical domain
  - A sequence of meshes $\rightsquigarrow \{\mathcal{T}_{\bullet\nu}\}_{\nu \in \mathfrak{P}_\bullet}$
  - $u_\bullet(x, \mathbf{y}) = \sum_{\nu \in \mathfrak{P}_\bullet} u_{\bullet\nu}(x) P_\nu(\mathbf{y})$ with $u_{\bullet\nu} \in \mathbb{X}_{\bullet\nu} = \mathcal{S}^1_0(\mathcal{T}_{\bullet\nu})$ for all $\nu \in \mathfrak{P}_\bullet$

Stochastic Galerkin FEM (2/2)

- $\{P_\nu : \nu \in \mathcal{I}\}$ is a countable orthonormal polynomial basis of $\mathbb{P} = L^2_\pi(\Gamma)$
- gPC expansion: $\mathbb{V} \ni u(x, \mathbf{y}) = \sum_{\nu \in \mathcal{I}} u_\nu(x) P_\nu(\mathbf{y})$ with unique $u_\nu \in \mathbb{X} = H^1_0(D)$
- $\mathcal{I} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \#\mathrm{supp}(\nu) < \infty\}$ where $\mathrm{supp}(\nu) = \{m \in \mathbb{N} : \nu_m \neq 0\}$
- Discretisation in the parameter domain
  - finite index set $\mathcal{P}_\bullet \subset \mathcal{I} \implies \mathbb{P}_\bullet = \mathrm{span}\{P_\nu : \nu \in \mathcal{P}_\bullet\} \subset \mathbb{P} = L^2_\pi(\Gamma)$
  - semidiscrete approximation via truncation of gPC expansion

$$u(x, \mathbf{y}) \approx \sum_{\nu \in \mathcal{P}_\bullet} u_\nu(x) P_\nu(\mathbf{y}) \in \mathbb{X} \otimes \mathbb{P}_\bullet \text{ with coefficients } u_\nu \in \mathbb{X} = H^1_0(D)$$

- Discretisations in the physical domain
  - A sequence of meshes $\rightsquigarrow \{\mathcal{T}_{\bullet\nu}\}_{\nu \in \mathcal{P}_\bullet}$
  - $u_\bullet(x, \mathbf{y}) = \sum_{\nu \in \mathcal{P}_\bullet} u_{\bullet\nu}(x) P_\nu(\mathbf{y})$ with $u_{\bullet\nu} \in \mathbb{X}_{\bullet\nu} = \mathcal{S}^1_0(\mathcal{T}_{\bullet\nu})$ for all $\nu \in \mathcal{P}_\bullet$
- $\mathbb{V}_\bullet := \bigoplus_{\nu \in \mathcal{P}_\bullet} \left[\mathbb{X}_{\bullet\nu} \otimes \mathrm{span}\{P_\nu\}\right]$, $\dim \mathbb{V}_\bullet = \sum_{\nu \in \mathcal{P}_\bullet} \dim \mathbb{X}_{\bullet\nu}$ (multilevel SGFEM)

Hierarchical a posteriori error estimators: main ideas

- Pythagoras theorem: $u \in \mathbb{V}$, $u_\bullet \in \mathbb{V}_\bullet \subset \mathbb{V}$, $\widehat{u}_\bullet \in \widehat{\mathbb{V}}_\bullet \supset \mathbb{V}_\bullet$ (enhanced approx.)

$$||| \, u - u_\bullet \, |||^2 = ||| \, (u - \widehat{u}_\bullet) + (\widehat{u}_\bullet - u_\bullet) \, |||^2 = ||| \, u - \widehat{u}_\bullet \, |||^2 + \underbrace{||| \, \widehat{u}_\bullet - u_\bullet \, |||^2}_{\text{computable}}$$

Hierarchical a posteriori error estimators: main ideas

- Pythagoras theorem: $u \in \mathbb{V}$, $u_\bullet \in \mathbb{V}_\bullet \subset \mathbb{V}$, $\widehat{u}_\bullet \in \widehat{\mathbb{V}}_\bullet \supset \mathbb{V}_\bullet$ (enhanced approx.)

$$|||\, u - u_\bullet \,|||^2 = |||\, (u - \widehat{u}_\bullet) + (\widehat{u}_\bullet - u_\bullet) \,|||^2 = |||\, u - \widehat{u}_\bullet \,|||^2 + \underbrace{|||\, \widehat{u}_\bullet - u_\bullet \,|||^2}_{\text{computable}}$$

$\Longrightarrow \quad |||\, u - u_\bullet \,||| \geq |||\, \widehat{u}_\bullet - u_\bullet \,|||$ (efficiency)

Hierarchical a posteriori error estimators: main ideas

- Pythagoras theorem: $u \in \mathbb{V}$, $u_\bullet \in \mathbb{V}_\bullet \subset \mathbb{V}$, $\widehat{u}_\bullet \in \widehat{\mathbb{V}}_\bullet \supset \mathbb{V}_\bullet$ (enhanced approx.)

$$|||\, u - u_\bullet \,|||^2 = |||\, (u - \widehat{u}_\bullet) + (\widehat{u}_\bullet - u_\bullet) \,|||^2 = |||\, u - \widehat{u}_\bullet \,|||^2 + \underbrace{|||\, \widehat{u}_\bullet - u_\bullet \,|||^2}_{\text{computable}}$$

$$\Longrightarrow \quad |||\, u - u_\bullet \,||| \geq |||\, \widehat{u}_\bullet - u_\bullet \,||| \quad \text{(efficiency)}$$

- Saturation assumption: $\exists\, q_{\rm sat} \in (0,1)$ such that $|||\, u - \widehat{u}_\bullet \,||| \leq q_{\rm sat} |||\, u - u_\bullet \,|||$

Hierarchical a posteriori error estimators: main ideas

- Pythagoras theorem: $u \in \mathbb{V}$, $u_\bullet \in \mathbb{V}_\bullet \subset \mathbb{V}$, $\widehat{u}_\bullet \in \widehat{\mathbb{V}}_\bullet \supset \mathbb{V}_\bullet$ (enhanced approx.)

$$|||\, u - u_\bullet \,|||^2 = |||\, (u - \widehat{u}_\bullet) + (\widehat{u}_\bullet - u_\bullet) \,|||^2 = |||\, u - \widehat{u}_\bullet \,|||^2 + \underbrace{|||\, \widehat{u}_\bullet - u_\bullet \,|||^2}_{\text{computable}}$$

$$\Longrightarrow \quad |||\, u - u_\bullet \,||| \geq |||\, \widehat{u}_\bullet - u_\bullet \,||| \qquad \text{(efficiency)}$$

- Saturation assumption: $\exists\, q_{\rm sat} \in (0,1)$ such that $|||\, u - \widehat{u}_\bullet \,||| \leq q_{\rm sat} |||\, u - u_\bullet \,|||$
- Pythagoras theorem & saturation assumption imply that

$$|||\, u - u_\bullet \,||| \leq (1 - q_{\rm sat}^2)^{-1/2}\, |||\, \widehat{u}_\bullet - u_\bullet \,||| \qquad \text{(reliability)}$$

Hierarchical a posteriori error estimators: main ideas

- Pythagoras theorem: $u \in \mathbb{V}$, $u_\bullet \in \mathbb{V}_\bullet \subset \mathbb{V}$, $\widehat{u}_\bullet \in \widehat{\mathbb{V}}_\bullet \supset \mathbb{V}_\bullet$ (enhanced approx.)

$$|||\, u - u_\bullet \,|||^2 = |||\, (u - \widehat{u}_\bullet) + (\widehat{u}_\bullet - u_\bullet) \,|||^2 = |||\, u - \widehat{u}_\bullet \,|||^2 + \underbrace{|||\, \widehat{u}_\bullet - u_\bullet \,|||^2}_{\text{computable}}$$

$$\Longrightarrow \quad |||\, u - u_\bullet \,||| \geq |||\, \widehat{u}_\bullet - u_\bullet \,||| \quad \text{(efficiency)}$$

- Saturation assumption: $\exists\, q_{\rm sat} \in (0,1)$ such that $|||\, u - \widehat{u}_\bullet \,||| \leq q_{\rm sat} |||\, u - u_\bullet \,|||$

- Pythagoras theorem & saturation assumption imply that

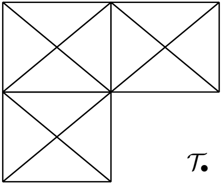
$$|||\, u - u_\bullet \,||| \leq (1 - q_{\rm sat}^2)^{-1/2} \,|||\, \widehat{u}_\bullet - u_\bullet \,||| \quad \text{(reliability)}$$

- [Bank, Weiser '85]: decomposition $\widehat{\mathbb{V}}_\bullet = \mathbb{V}_\bullet \oplus \mathbb{W}_\bullet$

  $\Longrightarrow$ hierarchical error estimation without computing enhanced approximations $\widehat{u}_\bullet$

Enhancement of SGFEM approximations (1/2)

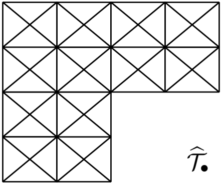
- Enhancement of approximations in physical domain
  - initial mesh $\mathcal{T}_0$
  - add new vertices to $\mathcal{T}_\bullet$ $\rightsquigarrow$ mesh refinement
  - mesh refinement by newest vertex bisection (NVB)
  - $\widehat{\mathcal{T}}_\bullet$ $\rightsquigarrow$ uniform refinement of $\mathcal{T}_\bullet$



$\mathcal{T}_\bullet$

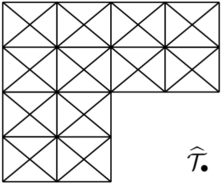
Enhancement of SGFEM approximations (1/2)

- Enhancement of approximations in physical domain
  - initial mesh $\mathcal{T}_0$
  - add new vertices to $\mathcal{T}_\bullet$ $\rightsquigarrow$ mesh refinement
  - mesh refinement by newest vertex bisection (NVB)
  - $\widehat{\mathcal{T}}_\bullet$ $\rightsquigarrow$ uniform refinement of $\mathcal{T}_\bullet$


$\widehat{\mathcal{T}}_\bullet$

Enhancement of SGFEM approximations (1/2)

- Enhancement of approximations in physical domain
  - initial mesh $\mathcal{T}_0$
  - add new vertices to $\mathcal{T}_\bullet$ $\rightsquigarrow$ mesh refinement
  - mesh refinement by newest vertex bisection (NVB)
  - $\widehat{\mathcal{T}}_\bullet$ $\rightsquigarrow$ uniform refinement of $\mathcal{T}_\bullet$



$\widehat{\mathcal{T}}_\bullet$

- Enhancement of approximations in the parameter domain
  - add new indices to $\mathcal{P}_\bullet$
  - finite set $\mathcal{Q}_\bullet \subset \mathcal{I} \setminus \mathcal{P}_\bullet$ $\rightsquigarrow$ detail index set ('boundary' of $\mathcal{P}_\bullet$)
  - $\widehat{\mathcal{P}}_\bullet = \mathcal{P}_\bullet \cup \mathcal{Q}_\bullet$ $\rightsquigarrow$ uniform enrichment of $\mathcal{P}_\bullet$
  - $\widehat{\mathbb{P}}_\bullet = \mathrm{span}\{P_\nu : \nu \in \widehat{\mathcal{P}}_\bullet\} \supset \mathbb{P}_\bullet$

Example

- $\mathcal{P}_\bullet = \{(0,0,\dots);(1,0,\dots);(0,1,0,\dots)\}$

  $\Longrightarrow \mathcal{Q}_\bullet = \{(2,0,\dots);(1,1,0,\dots);(0,2,0,\dots);(0,0,1,0,\dots)\}$

Enhancement of SGFEM approximation (2/2)

- Two sources of error: $|||\, u - u_\bullet \,|||^2 = (\mathbb{X}\text{-errors})^2 + (\mathbb{P}\text{-error})^2$

Enhancement of SGFEM approximation (2/2)

- Two sources of error: $\| u - u_\bullet \|^2 = (\mathbb{X}\text{-errors})^2 + (\mathbb{P}\text{-error})^2$
- Choice of $\widehat{\mathbb{V}}_\bullet$ such that $\| u - u_\bullet \| \simeq \| \widehat{u}_\bullet - u_\bullet \|$ with $\widehat{u}_\bullet \in \widehat{V}_\bullet$?

Enhancement of SGFEM approximation (2/2)

- Two sources of error: $\| u - u_\bullet \|^2 = (\mathbb{X}\text{-errors})^2 + (\mathbb{P}\text{-error})^2$
- Choice of $\widehat{\mathbb{V}}_\bullet$ such that $\| u - u_\bullet \| \simeq \| \widehat{u}_\bullet - u_\bullet \|$ with $\widehat{u}_\bullet \in \widehat{V}_\bullet$?
- Enhanced approximation space $\widehat{\mathbb{V}}_\bullet$

  [B., Praetorius, Ruggeri; SIAM/ASA JUQ '21]

$$\widehat{\mathbb{V}}_\bullet = \underbrace{\bigoplus_{\nu \in \mathcal{P}_\bullet} [\widehat{\mathbb{X}}_{\bullet\nu} \otimes \mathrm{span}\{P_\nu\}]}_{\text{spatial enhancement}} \oplus \underbrace{\bigoplus_{\nu \in \mathcal{Q}_\bullet} [\mathbb{X}_0 \otimes \mathrm{span}\{P_\nu\}]}_{\text{parametric enhancement}}$$

Enhancement of SGFEM approximation (2/2)

- Two sources of error: $|||\, u - u_\bullet \,|||^2 = (\mathbb{X}\text{-errors})^2 + (\mathbb{P}\text{-error})^2$
- Choice of $\widehat{\mathbb{V}}_\bullet$ such that $|||\, u - u_\bullet \,||| \simeq |||\, \widehat{u}_\bullet - u_\bullet \,|||$ with $\widehat{u}_\bullet \in \widehat{V}_\bullet$?
- Enhanced approximation space $\widehat{\mathbb{V}}_\bullet$

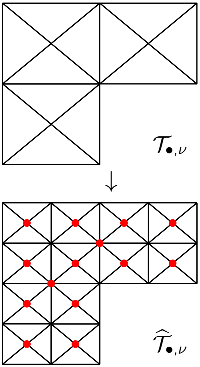
  [B., Praetorius, Ruggeri; SIAM/ASA JUQ '21]

$$\widehat{\mathbb{V}}_\bullet = \underbrace{\bigoplus_{\nu \in \mathcal{P}_\bullet} [\widehat{\mathbb{X}}_{\bullet\nu} \otimes \mathrm{span}\{P_\nu\}]}_{\text{spatial enhancement}} \oplus \underbrace{\bigoplus_{\nu \in \mathcal{Q}_\bullet} [\mathbb{X}_0 \otimes \mathrm{span}\{P_\nu\}]}_{\text{parametric enhancement}}$$

- Recap: $|||\, u - u_\bullet \,||| \simeq |||\, \widehat{u}_\bullet - u_\bullet \,|||$, but we want to avoid computing $\widehat{u}_\bullet \in \widehat{V}_\bullet$!

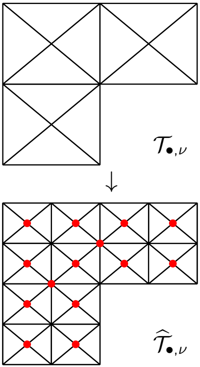
A posteriori error estimation: spatial and parametric estimators

- Two-level spatial error estimator
  [Mund, Stephan, Weiße '98], [Mund, Stephan '99]
  - Fix a multiindex $\nu \in \mathcal{P}_\bullet$
  - $\mathcal{N}^+_{\bullet,\nu} \rightsquigarrow$ set of interior midpoints in $\mathcal{T}_{\bullet,\nu}$
  - $\widehat{\varphi}_{\bullet,\nu,z} \in \widehat{\mathbb{X}}_{\bullet,\nu} \rightsquigarrow$ hat function associated with $z \in \mathcal{N}^+_{\bullet,\nu}$
  - $\eta^2_\bullet(\nu, z) = \dfrac{|F(\widehat{\varphi}_{\bullet,\nu,z} P_\nu) - B(u_\bullet, \widehat{\varphi}_{\bullet,\nu,z} P_\nu)|^2}{\|a_0^{1/2} \nabla \widehat{\varphi}_{\bullet,\nu,z}\|^2_{L^2(D)}}$
  - $(\mathbb{X}\text{-errors})^2 \approx \sum_{\nu \in \mathcal{P}_\bullet} \sum_{z \in \mathcal{N}^+_{\bullet,\nu}} \eta^2_\bullet(\nu, z)$



A posteriori error estimation: spatial and parametric estimators

- Two-level spatial error estimator
  [Mund, Stephan, Weiße '98], [Mund, Stephan '99]
  - Fix a multiindex $\nu \in \mathcal{P}_\bullet$
  - $\mathcal{N}^+_{\bullet,\nu} \rightsquigarrow$ set of interior midpoints in $\mathcal{T}_{\bullet,\nu}$
  - $\widehat{\varphi}_{\bullet,\nu,z} \in \widehat{\mathbb{X}}_{\bullet,\nu} \rightsquigarrow$ hat function associated with $z \in \mathcal{N}^+_{\bullet,\nu}$
  - $\eta^2_\bullet(\nu, z) = \dfrac{|F(\widehat{\varphi}_{\bullet,\nu,z} P_\nu) - B(u_\bullet, \widehat{\varphi}_{\bullet,\nu,z} P_\nu)|^2}{\|a_0^{1/2} \nabla \widehat{\varphi}_{\bullet,\nu,z}\|^2_{L^2(D)}}$
  - $(\mathbb{X}\text{-errors})^2 \approx \sum_{\nu \in \mathcal{P}_\bullet} \sum_{z \in \mathcal{N}^+_{\bullet,\nu}} \eta^2_\bullet(\nu, z)$



- Hierarchical parametric error estimator
  [B., Silvester '16], [B., Praetorius, Ruggeri; SIAM/ASA JUQ '21]
  - $(\mathbb{P}\text{-error})^2 \approx \sum_{\nu \in \mathcal{Q}_\bullet} \eta^2_\bullet(\nu)$

A posteriori error estimation: main results

[B., Praetorius, Ruggeri; SIAM/ASA JUQ '21]

- $\eta_\bullet^2 = (\text{estim. } \mathbb{X}\text{-errors})^2 + (\text{estim. } \mathbb{P}\text{-error})^2 = \sum_{\nu \in \mathcal{P}_\bullet} \sum_{z \in \mathcal{N}_{\bullet,\nu}^+} \eta_\bullet^2(\nu, z) + \sum_{\nu \in \mathcal{Q}_\bullet} \eta_\bullet^2(\nu)$

Theorem 1 (equivalence of total error estimate and error reduction)

There exists $C = C(a_0, \tau, \mathcal{T}_0) \geq 1$ such that

$$C^{-1}\,\eta_\bullet^2 \leq |||\,\widehat{u}_\bullet - u_\bullet\,|||^2 \leq C\,\eta_\bullet^2$$

A posteriori error estimation: main results

[B., Praetorius, Ruggeri; SIAM/ASA JUQ '21]

- $\eta_\bullet^2 = (\text{estim. } \mathbb{X}\text{-errors})^2 + (\text{estim. } \mathbb{P}\text{-error})^2 = \sum_{\nu \in \mathcal{P}_\bullet} \sum_{z \in \mathcal{N}_{\bullet,\nu}^+} \eta_\bullet^2(\nu, z) + \sum_{\nu \in \mathcal{Q}_\bullet} \eta_\bullet^2(\nu)$

Theorem 1 (equivalence of total error estimate and error reduction)

There exists $C = C(a_0, \tau, \mathcal{T}_0) \geq 1$ such that

$$C^{-1}\,\eta_\bullet^2 \leq |||\,\widehat{u}_\bullet - u_\bullet\,|||^2 \leq C\,\eta_\bullet^2$$

Corollary (efficiency & reliability)

- $|||\,u - u_\bullet\,|||^2 \geq C^{-1}\,\eta_\bullet^2$ (efficiency)
- saturation assumption $\implies$ $|||\,u - u_\bullet\,|||^2 \leq \dfrac{C}{1 - q_{\mathrm{sat}}^2}\,\eta_\bullet^2$ (reliability)

A posteriori error estimation: main results

[B., Praetorius, Ruggeri; SIAM/ASA JUQ '21]

- $\eta_\bullet^2 = (\text{estim. } \mathbb{X}\text{-errors})^2 + (\text{estim. } \mathbb{P}\text{-error})^2 = \sum_{\nu \in \mathfrak{P}_\bullet} \sum_{z \in \mathcal{N}_{\bullet,\nu}^+} \eta_\bullet^2(\nu, z) + \sum_{\nu \in \mathfrak{Q}_\bullet} \eta_\bullet^2(\nu)$

Theorem 1 (equivalence of total error estimate and error reduction)

There exists $C = C(a_0, \tau, \mathcal{T}_0) \geq 1$ such that

$$C^{-1}\, \eta_\bullet^2 \leq |||\, \widehat{u}_\bullet - u_\bullet \,|||^2 \leq C\, \eta_\bullet^2$$

Remark

- $\eta_\bullet(\nu, z)$ is associated with an interior edge midpoint $z \in \mathcal{N}_{\bullet,\nu}^+$ for each $\nu \in \mathfrak{P}_\bullet$
- $\eta_\bullet(\nu)$ is associated with a new index $\nu \in \mathfrak{Q}_\bullet$

These are local *error reduction indicators* for
spatial refinement / parametric enrichment $\Longrightarrow$ key to adaptivity

Adaptive SGFEM algorithm

INPUT: initial mesh $\mathcal{T}_0$, initial index set $\mathcal{P}_0 = \{(0, 0, \dots)\}$, tolerance tol

FOR $\ell = 0, 1, 2, 3, \dots$ DO:

- SOLVE: compute $u_\ell \in \mathbb{V}_\ell$ for index set $\mathcal{P}_\ell$ and meshes $\mathcal{T}_{\ell,\nu}$ $(\nu \in \mathcal{P}_\ell)$
- ESTIMATE: compute *local* error indicators and the *total* error estimate
  - spatial & parametric indicators
    $\{\eta_\ell(\nu, z);\ z \in \mathcal{N}^+_{\ell,\nu},\ \nu \in \mathcal{P}_\ell\}$ & $\{\eta_\ell(\nu);\ \nu \in \mathcal{Q}_\ell\}$
  - energy error estimate η_ℓ
  - IF $\eta_\ell <$ tol THEN STOP
- MARK: mark certain vertices $\mathcal{M}_{\ell,\nu} \subseteq \mathcal{N}^+_{\ell,\nu}$ $(\nu \in \mathcal{P}_\ell)$ and indices $\mathcal{R}_\ell \subseteq \mathcal{Q}_\ell$
- REFINE: enhance approximation space
  - mesh refinement (NVB) $\rightsquigarrow$ $\mathcal{T}_{\ell+1,\nu} = \mathbf{refine}(\mathcal{T}_{\ell,\nu}, \mathcal{M}_{\ell,\nu})\ \ \forall \nu \in \mathcal{P}_\ell$
  - parametric enrichment $\rightsquigarrow$ $\mathcal{P}_{\ell+1} = \mathcal{P}_\ell \cup \mathcal{R}_\ell$, $\mathcal{T}_{\ell,\nu} = \mathcal{T}_0\ \ \forall \nu \in \mathcal{R}_\ell$

OUTPUT: stochastic Galerkin approximations $\{u_\ell\}$ and error estimates $\{\eta_\ell\}$

Convergence results

[B., Praetorius, Rocchi, Ruggeri; SINUM '19]

[B., Praetorius, Ruggeri; arXiv preprint '22]

Theorem 2 (plain convergence)

For any marking threshold $\theta \in (0, 1]$ (in Dörfler marking), adaptive multilevel SGFEM algorithm yields a convergent sequence of error estimates, i.e., $\eta_\ell \to 0$ as $\ell \to \infty$.

Convergence results

[B., Praetorius, Rocchi, Ruggeri; SINUM '19]

[B., Praetorius, Ruggeri; arXiv preprint '22]

Theorem 2 (plain convergence)

For any marking threshold $\theta \in (0, 1]$ (in Dörfler marking), adaptive multilevel SGFEM algorithm yields a convergent sequence of error estimates, i.e., $\eta_\ell \to 0$ as $\ell \to \infty$.

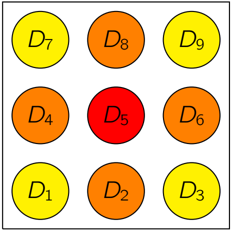
Theorem 3 (linear convergence)

For any marking threshold $\theta \in (0, 1]$, saturation assumption $\Longrightarrow$ linear convergence

$$\exists\, q \in (0, 1) \;\text{ such that }\; |\!|\!| u - u_{\ell+1} |\!|\!| \le q\, |\!|\!| u - u_\ell |\!|\!| \quad \text{for all } \ell \in \mathbb{N}_0$$

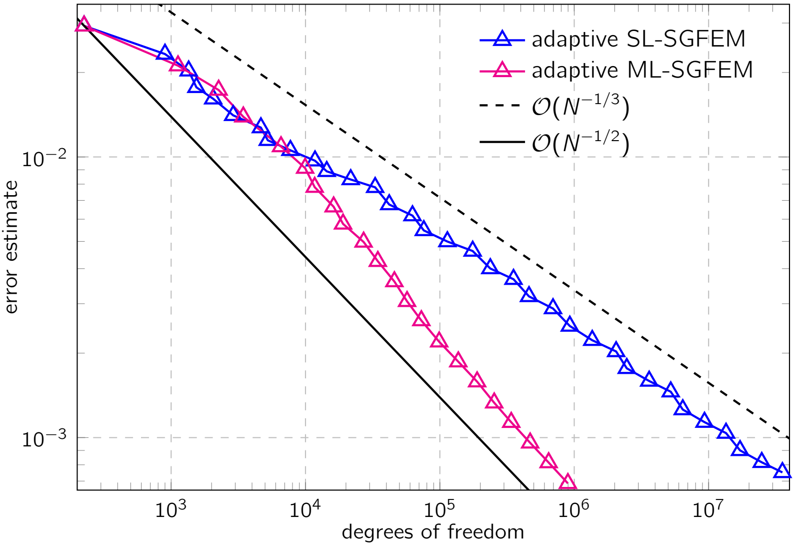
Experiment 1: cookie problem

- $-\nabla \cdot (a \nabla u) = f$ in $D \times \Gamma$, $\quad u = 0$ on $\partial D \times \Gamma$
- $a(x, \mathbf{y}) = a_0(x) + \sum_{m \in \mathbb{N}} y_m a_m(x)$
- $D = (0,1)^2 \rightsquigarrow$ square domain
  - nine circular inclusions $D_m \subset D$ $(m = 1, \ldots, 9)$
- Expansion coefficients $\{a_m\}_{m \in \mathbb{N}_0}$
  - $a_0 \equiv 1$
  - $a_m = 0.5\,\chi_{D_m}$ for $m = 1, 3, 7, 9$
  - $a_m = 0.7\,\chi_{D_m}$ for $m = 2, 4, 6, 8$
  - $a_m = 0.9\,\chi_{D_m}$ for $m = 5$
  - $a_m \equiv 0$ for $m > 9$
- $f \equiv 1$
- $\mathrm{d}\pi_m(y_m) = \frac{1}{2}\,\mathrm{d}y_m \rightsquigarrow$ uniform probability measure on $[-1, 1]$

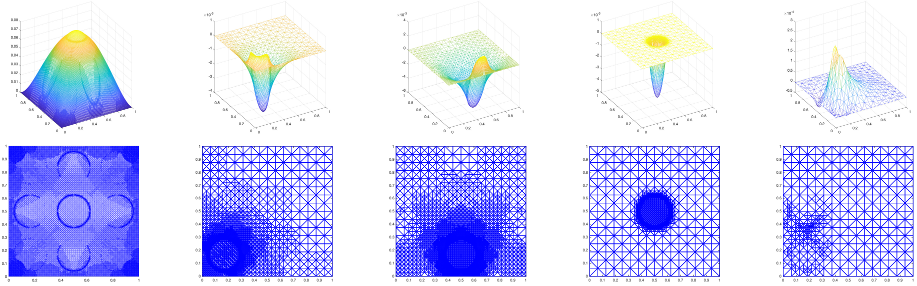


D

Experiment 1: rate optimality of adaptive ML-SGFEM



Experiment 1: locally refined meshes in ML-SGFEM



$\nu = (0\,0\ldots0)$	$\nu = (1\,0\ldots0)$	$\nu = (0\,1\,0\ldots0)$	ν=(0 0 0 0 1 0...0)	$\nu = (1\,0\,0\,1\,0\ldots0)$
$\#\mathcal{T}_{\ell,\nu} = 84050$	$\#\mathcal{T}_{\ell,\nu} = 10994$	$\#\mathcal{T}_{\ell,\nu} = 16420$	$\#\mathcal{T}_{\ell,\nu} = 9528$	$\#\mathcal{T}_{\ell,\nu} = 839$

Optimal convergence of adaptive multilevel SGFEM

[B., Praetorius, Ruggeri; IMA J. Numer. Anal. '22]

- Concept of 'multilevel structure' $\rightsquigarrow$ $\mathbb{P}_\bullet = [\mathcal{P}_\bullet, (\mathcal{T}_{\bullet\nu})_{\nu \in \mathcal{P}_\bullet}]$, $\#\mathbb{P}_\bullet \simeq \dim \mathbb{V}_\bullet$
- Concept of 'multilevel refinement' $\rightsquigarrow$ $\mathbb{P}_\circ = \mathbb{REFINE}(\mathbb{P}_\bullet, \mathbb{M}_\bullet)$
- Concept of optimality $\rightsquigarrow$ approximation class $\mathbb{A}_s$ $(s > 0)$

$$u \in \mathbb{A}_s \quad \Longleftrightarrow \quad \exists \, \{\mathbb{P}_\ell^\star\}_{\ell \in \mathbb{N}_0} \text{ such that } \| u - u_\ell^\star \| = \mathcal{O}\big((\dim \mathbb{V}_\ell^\star)^{-s}\big)$$

Optimal convergence of adaptive multilevel SGFEM

[B., Praetorius, Ruggeri; IMA J. Numer. Anal. '22]

- Concept of 'multilevel structure' $\rightsquigarrow \mathbb{P}_\bullet = [\mathcal{P}_\bullet, (\mathcal{T}_{\bullet\nu})_{\nu\in\mathcal{P}_\bullet}]$, $\#\mathbb{P}_\bullet \simeq \dim \mathbb{V}_\bullet$
- Concept of 'multilevel refinement' $\rightsquigarrow \mathbb{P}_\circ = \mathbb{REFINE}(\mathbb{P}_\bullet, \mathbb{M}_\bullet)$
- Concept of optimality $\rightsquigarrow$ approximation class $\mathbb{A}_s$ $(s > 0)$

$$u \in \mathbb{A}_s \quad \Longleftrightarrow \quad \exists \left\{\mathbb{P}_\ell^\star\right\}_{\ell\in\mathbb{N}_0} \text{ such that } |||\, u - u_\ell^\star \,||| = \mathcal{O}\Big((\dim \mathbb{V}_\ell^\star)^{-s}\Big)$$

Theorem 4 (rate optimality of adaptive multilevel SGFEM)

For sufficiently small marking threshold θ, strong saturation assumption $\Longrightarrow$ optimal convergence

$$\text{If } s > 0 \text{ and } u \in \mathbb{A}_s, \text{ then } \sup_{\ell\in\mathbb{N}_0} (\#\mathbb{P}_\ell - \#\mathbb{P}_0 + 1)^s \, |||\, u - u_\ell \,||| \le C \|u\|_{\mathbb{A}_s}$$

Saturation assumption vs. strong saturation assumption

- Saturation assumption
  - SGFEM solution $u_\bullet \in \mathbb{V}_\bullet$
  - Enhanced ('uniformly refined') SGFEM solution $\widehat{u}_\bullet \in \widehat{\mathbb{V}}_\bullet$
  - There exist a constant $q_{\mathrm{sat}} \in (0,1)$ s.t. $|||\, u - \widehat{u}_\bullet \,||| \le q_{\mathrm{sat}} |||\, u - u_\bullet \,|||$

Saturation assumption vs. strong saturation assumption

- Saturation assumption
  - SGFEM solution $u_\bullet \in \mathbb{V}_\bullet$
  - Enhanced ('uniformly refined') SGFEM solution $\widehat{u}_\bullet \in \widehat{\mathbb{V}}_\bullet$
  - There exist a constant $q_{\rm sat} \in (0,1)$ s.t. $|||\, u - \widehat{u}_\bullet \,||| \le q_{\rm sat} |||\, u - u_\bullet \,|||$
- Strong saturation assumption
  - $\mathbb{P}_\bullet \in \mathbb{REFINE}(\mathbb{P}_0)$ $\rightsquigarrow$ any multilevel structure obtained from $\mathbb{P}_0$
  - $\mathbb{P}_\star \in \mathbb{REFINE}(\mathbb{P}_\bullet)$ $\rightsquigarrow$ a refined multilevel structure obtained from $\mathbb{P}_\bullet$
  - $\mathbb{P}_\circ := \mathbb{REFINE}(\mathbb{P}_\bullet, \mathbb{M}_\bullet)$ $\rightsquigarrow$ a multilevel structure obtained from $\mathbb{P}_\bullet$ by one step of multilevel refinement towards $\mathbb{P}_\star$
  - There exist constants $\exists\, 0 < \kappa_{\rm sat} \le q_{\rm sat} < 1$ such that

$$|||\, u - u_\star \,||| \le \kappa_{\rm sat} \,|||\, u - u_\bullet \,||| \quad \Longrightarrow \quad |||\, u - u_\circ \,||| \le q_{\rm sat} \,|||\, u - u_\bullet \,|||$$

Parametric model problem revisited

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) = f(x,\mathbf{y}) \qquad x \in D,\ \mathbf{y} \in \Gamma,$$
$$u(x,\mathbf{y}) = 0 \qquad x \in \partial D,\ \mathbf{y} \in \Gamma$$

- Parametric diffusion coefficient
  - $\Gamma := [-1,1]^M \rightsquigarrow$ parameter domain, $M \in \mathbb{N}$
  - $0 < a_{\min} \leq \operatorname*{ess\,inf}_{x\in D} a(x,\mathbf{y}) \leq \operatorname*{ess\,sup}_{x\in D} a(x,\mathbf{y}) \leq a_{\max} < \infty \quad \pi\text{-a.e. on } \Gamma$

Parametric model problem revisited

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) = f(x,\mathbf{y}) \qquad x \in D,\ \mathbf{y} \in \Gamma,$$
$$u(x,\mathbf{y}) = 0 \qquad x \in \partial D,\ \mathbf{y} \in \Gamma$$

- Parametric diffusion coefficient
  - $\Gamma := [-1,1]^M \rightsquigarrow$ parameter domain, $M \in \mathbb{N}$
  - $0 < a_{\min} \le \operatorname*{ess\,inf}_{x\in D} a(x,\mathbf{y}) \le \operatorname*{ess\,sup}_{x\in D} a(x,\mathbf{y}) \le a_{\max} < \infty \quad \pi$-a.e. on Γ
- Weak formulation: given $f \in L^2_\pi(\Gamma, L^2(D))$, find $u : \Gamma \to \mathbb{X} := H^1_0(D)$ s.t.

$$\int_D a(x,\mathbf{y})\nabla u(x,\mathbf{y}) \cdot \nabla v(x)\,\mathrm{d}x = \int_D f(x,\mathbf{y})v(x)\,\mathrm{d}x \quad \forall v \in \mathbb{X},\ \pi\text{-a.e. on } \Gamma.$$

Parametric model problem revisited

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

$$-\nabla_x \cdot (a(x, \mathbf{y}) \nabla_x u(x, \mathbf{y})) = f(x, \mathbf{y}) \qquad x \in D, \ \mathbf{y} \in \Gamma,$$
$$u(x, \mathbf{y}) = 0 \qquad x \in \partial D, \ \mathbf{y} \in \Gamma$$

- Parametric diffusion coefficient
  - $\Gamma := [-1, 1]^M \rightsquigarrow$ parameter domain, $M \in \mathbb{N}$
  - $0 < a_{\min} \le \operatorname*{ess\,inf}_{x \in D} a(x, \mathbf{y}) \le \operatorname*{ess\,sup}_{x \in D} a(x, \mathbf{y}) \le a_{\max} < \infty \quad \pi$-a.e. on Γ
- Weak formulation: given $f \in L^2_\pi(\Gamma, L^2(D))$, find $u : \Gamma \to \mathbb{X} := H^1_0(D)$ s.t.

$$\int_D a(x, \mathbf{y}) \nabla u(x, \mathbf{y}) \cdot \nabla v(x) \, \mathrm{d}x = \int_D f(x, \mathbf{y}) v(x) \, \mathrm{d}x \quad \forall v \in \mathbb{X}, \ \pi\text{-a.e. on } \Gamma.$$

- [Babuška, Nobile, Tempone '07]: $\exists! \, u \in \mathbb{V} := L^2_\pi(\Gamma; \mathbb{X})$.

Sparse grid stochastic collocation FEM

- Sampling the PDE inputs at a sparse grid $\mathcal{Y}_\bullet = \mathcal{Y}_{\Lambda_\bullet}$ of collocation points in Γ

Sparse grid stochastic collocation FEM

- Sampling the PDE inputs at a sparse grid $\mathcal{Y}_\bullet = \mathcal{Y}_{\Lambda_\bullet}$ of collocation points in Γ
- Solving decoupled discrete problems: for each $\mathbf{z} \in \mathcal{Y}_\bullet$, find $u_{\bullet\mathbf{z}} \in \mathbb{X}_{\bullet\mathbf{z}}$ satisfying

$$\int_D a(x,\mathbf{z})\nabla u_{\bullet\mathbf{z}}(x)\cdot\nabla v(x)\mathrm{d}x = \int_D f(x,\mathbf{z})v(x)\mathrm{d}x \;\; \forall v \in \mathbb{X}_{\bullet\mathbf{z}} := \mathcal{S}_0^1(\mathcal{T}_{\bullet\mathbf{z}}) \subset H_0^1(D)$$

- Sampling the PDE inputs at a sparse grid $\mathcal{Y}_\bullet = \mathcal{Y}_{\Lambda_\bullet}$ of collocation points in Γ
- Solving decoupled discrete problems: for each $\mathbf{z} \in \mathcal{Y}_\bullet$, find $u_{\bullet\mathbf{z}} \in \mathbb{X}_{\bullet\mathbf{z}}$ satisfying

$$\int_D a(x,\mathbf{z})\nabla u_{\bullet\mathbf{z}}(x)\cdot\nabla v(x)\mathrm{d}x = \int_D f(x,\mathbf{z})v(x)\mathrm{d}x \;\; \forall v \in \mathbb{X}_{\bullet\mathbf{z}} := \mathcal{S}_0^1(\mathcal{T}_{\bullet\mathbf{z}}) \subset H_0^1(D)$$

- Building a multivariable interpolant

$$u_\bullet^{\mathrm{SC}}(x,\mathbf{y}) = \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x)L_{\bullet\mathbf{z}}(\mathbf{y}),$$

$\{L_{\bullet\mathbf{z}}(\mathbf{y}) : \mathbf{z} \in \mathcal{Y}_\bullet\}$ – multivariable Lagrange basis functions associated with $\mathcal{Y}_\bullet$

Sparse grid stochastic collocation FEM

- Sampling the PDE inputs at a sparse grid $\mathcal{Y}_\bullet = \mathcal{Y}_{\Lambda_\bullet}$ of collocation points in Γ
- Solving decoupled discrete problems: for each $\mathbf{z} \in \mathcal{Y}_\bullet$, find $u_{\bullet\mathbf{z}} \in \mathbb{X}_{\bullet\mathbf{z}}$ satisfying

$$\int_D a(x,\mathbf{z})\nabla u_{\bullet\mathbf{z}}(x)\cdot\nabla v(x)\mathrm{d}x = \int_D f(x,\mathbf{z})v(x)\mathrm{d}x \ \ \forall v \in \mathbb{X}_{\bullet\mathbf{z}} := \mathcal{S}_0^1(\mathcal{T}_{\bullet\mathbf{z}}) \subset H_0^1(D)$$

- Building a multivariable interpolant

$$u_\bullet^{\mathrm{SC}}(x,\mathbf{y}) = \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x)L_{\bullet\mathbf{z}}(\mathbf{y}),$$

$\{L_{\bullet\mathbf{z}}(\mathbf{y}) : \mathbf{z} \in \mathcal{Y}_\bullet\}$ – multivariable Lagrange basis functions associated with $\mathcal{Y}_\bullet$

- Main features of the stochastic collocation FEM (SC-FEM)
  - a sampling method that generates 'surrogate approximations'
  - single-level ($\mathbb{X}_{\bullet\mathbf{z}} = \mathbb{X}_\bullet \ \forall\, \mathbf{z} \in \mathcal{Y}_\bullet$) *vs.* multilevel ($\mathbb{X}_{\bullet\mathbf{z}} \neq \mathbb{X}_{\bullet\mathbf{z}'}$ for $\mathbf{z} \neq \mathbf{z}'$)
  - not a projection method $\rightsquigarrow$ no (global) Galerkin orthogonality

Hierarchical a posteriori error estimation in SC-FEM (1/2)

[B., Silvester, Xu '22], [B., Silvester '23]

- $\mathbb{V} := L^2_\pi(\Gamma; H^1_0(D))$, $\|\cdot\| := \|\cdot\|_{\mathbb{V}}$
- An enhanced SC-FEM approximation $\widehat{u}_\bullet^{\mathrm{SC}}$ satisfying the saturation property

$$\|u - \widehat{u}_\bullet^{\mathrm{SC}}\| \le q_{\mathrm{sat}} \|u - u_\bullet^{\mathrm{SC}}\| \quad \text{with } q_{\mathrm{sat}} \in (0,1)$$

- This gives a reliable error estimate

$$\|u - u_\bullet^{\mathrm{SC}}\| \le \big(1 - q_{\mathrm{sat}}\big)^{-1} \|\widehat{u}_\bullet^{\mathrm{SC}} - u_\bullet^{\mathrm{SC}}\|$$

- How to choose the enhanced approximation $\widehat{u}_\bullet^{\mathrm{SC}}$?

Hierarchical a posteriori error estimation in SC-FEM (2/2)

Recall that $u_\bullet^{\mathrm{SC}}(x,\mathbf{y}) = \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x) L_{\bullet\mathbf{z}}(\mathbf{y})$

- $$\widehat{u}_\bullet^{\mathrm{SC}}(x,\mathbf{y}) := \underbrace{\sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \widehat{u}_{\bullet\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y})}_{\text{spatial enhancement}} + \underbrace{\left(\sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet} u_{0\mathbf{z}'}(x)\,\widehat{L}_{\bullet\mathbf{z}'}(\mathbf{y}) - \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{0\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y})\right)}_{\text{parametric enhancement}}$$
  - $\widehat{u}_{\bullet\mathbf{z}} \in \widehat{\mathbb{X}}_{\bullet\mathbf{z}}$ (uniform mesh-refinement) $\forall\, \mathbf{z} \in \mathcal{Y}_\bullet = \mathcal{Y}_{\Lambda_\bullet}$
  - $\widehat{\mathcal{Y}}_\bullet = \mathcal{Y}_{\widehat{\Lambda}_\bullet}$ with $\widehat{\Lambda}_\bullet := \Lambda_\bullet \cup \mathrm{R}(\Lambda_\bullet)$ $\rightsquigarrow$ $\widehat{\Lambda}_\bullet$ is monotone!
  - $u_{0\mathbf{z}}(x),\ u_{0\mathbf{z}'}(x) \in \mathbb{X}_0 := \mathcal{S}_0^1(\mathcal{T}_0)$ $\forall\, \mathbf{z}\in\mathcal{Y}_\bullet$ and $\forall\, \mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet$

Hierarchical a posteriori error estimation in SC-FEM (2/2)

Recall that $u_\bullet^{\rm SC}(x,\mathbf{y}) = \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x) L_{\bullet\mathbf{z}}(\mathbf{y})$

- $$\widehat{u}_\bullet^{\rm SC}(x,\mathbf{y}) := \underbrace{\sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \widehat{u}_{\bullet\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y})}_{\text{spatial enhancement}} + \underbrace{\left(\sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet} u_{0\mathbf{z}'}(x)\, \widehat{L}_{\bullet\mathbf{z}'}(\mathbf{y}) - \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{0\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y}) \right)}_{\text{parametric enhancement}}$$

- A posteriori error estimate

$$\|u - u_\bullet^{\rm SC}\| \le \frac{1}{1-q_{\rm sat}} \|\widehat{u}_\bullet^{\rm SC} - u_\bullet^{\rm SC}\|$$

$$\le \frac{1}{1-q_{\rm sat}} \Bigg(\underbrace{\Big\| \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} (\widehat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}})\, L_{\bullet\mathbf{z}} \Big\|}_{\text{spatial estimate}} + \underbrace{\Big\| \sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet\setminus\mathcal{Y}_\bullet} \Big(u_{0\mathbf{z}'} - \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{0\mathbf{z}} L_{\bullet\mathbf{z}}(\mathbf{z}') \Big) \widehat{L}_{\bullet\mathbf{z}'} \Big\|}_{\text{parametric estimate}} \Bigg)$$

Hierarchical a posteriori error estimation in SC-FEM (2/2)

Recall that $u_\bullet^{\mathrm{SC}}(x,\mathbf{y}) = \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x) L_{\bullet\mathbf{z}}(\mathbf{y})$

- $$\widehat{u}_\bullet^{\mathrm{SC}}(x,\mathbf{y}) := \underbrace{\sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \widehat{u}_{\bullet\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y})}_{\text{spatial enhancement}} + \underbrace{\left(\sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet} u_{0\mathbf{z}'}(x)\, \widehat{L}_{\bullet\mathbf{z}'}(\mathbf{y}) - \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{0\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y}) \right)}_{\text{parametric enhancement}}$$

- A posteriori error estimate

$$\|u - u_\bullet^{\mathrm{SC}}\| \le \frac{1}{1-q_{\mathrm{sat}}} \|\widehat{u}_\bullet^{\mathrm{SC}} - u_\bullet^{\mathrm{SC}}\|$$

$$\le \frac{1}{1-q_{\mathrm{sat}}} \Bigg(\underbrace{\Big\| \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} (\widehat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}})\, L_{\bullet\mathbf{z}} \Big\|}_{\text{spatial estimate}} + \underbrace{\Big\| \sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet\setminus\mathcal{Y}_\bullet} \Big(u_{0\mathbf{z}'} - \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{0\mathbf{z}} L_{\bullet\mathbf{z}}(\mathbf{z}') \Big) \widehat{L}_{\bullet\mathbf{z}'} \Big\|}_{\text{parametric estimate}} \Bigg)$$

- Error indicators (e.g., spatial error indicators $\mu_{\bullet\mathbf{z}}$)

$$\mu_\bullet := \Big\| \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} (\widehat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}})\, L_{\bullet\mathbf{z}} \Big\| \le \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \|\widehat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}}\|_{\mathbb{X}} \, \|L_{\bullet\mathbf{z}}\|_{L^2_\pi(\Gamma)} \lesssim \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \mu_{\bullet\mathbf{z}} \, \|L_{\bullet\mathbf{z}}\|_{L^2_\pi(\Gamma)}$$

Adaptive multilevel SC-FEM algorithm

INPUT: $\Lambda_0 = \{\mathbf{1}\}$; $\mathcal{T}_{0\mathbf{z}} := \mathcal{T}_0\ \forall\, \mathbf{z} \in \widehat{\mathcal{Y}}_0 = \mathcal{Y}_{\Lambda_0 \cup \mathrm{R}(\Lambda_0)}$; output counter k; tolerance `tol`

FOR $\ell = 0, 1, 2, 3, \ldots$ DO:

- SOLVE: compute $u_{\ell\mathbf{z}} \in \mathbb{X}_{\ell\mathbf{z}}$ for all $\mathbf{z} \in \widehat{\mathcal{Y}}_\ell = \mathcal{Y}_{\widehat{\Lambda}_\ell} = \mathcal{Y}_{\Lambda_\ell \cup \mathrm{R}(\Lambda_\ell)}$
- ESTIMATE: compute error indicators
  - spatial indicators $\mu_{\ell\mathbf{z}}$ for all $\mathbf{z} \in \mathcal{Y}_\ell$
  - parametric indicators $\tau_{\ell\boldsymbol{\nu}}$ for all $\boldsymbol{\nu} \in \mathrm{R}(\Lambda_\ell)$
  - If $\ell = jk$, $j \in \mathbb{N}$, compute the total error estimate η_ℓ and exit if $\eta_\ell <$ `tol`
- MARK: mark certain edges/elements $\mathcal{M}_{\ell\mathbf{z}}$ ($\mathbf{z} \in \mathcal{Y}_\ell$) and indices $\Upsilon_\ell \subseteq \mathrm{R}(\Lambda_\ell)$
- REFINE: enhance approximations
  - mesh refinement (NVB) $\rightsquigarrow$ $\mathcal{T}_{(\ell+1)\mathbf{z}} := \mathbf{refine}(\mathcal{T}_{\ell\mathbf{z}}, \mathcal{M}_{\ell\mathbf{z}})$ for all $\mathbf{z} \in \mathcal{Y}_\ell$
  - parametric enrichment $\rightsquigarrow$ $\Lambda_{\ell+1} := \Lambda_\ell \cup \Upsilon_\ell$ and construct meshes $\mathcal{T}_{(\ell+1)\mathbf{z}'}$ for each new collocation point $\mathbf{z}'$

OUTPUT: SC-FEM approximation $u_{\ell^*}^{\mathrm{SC}}$ and the error estimate η_{ℓ^*} for some $\ell^* = jk$

Parametric enrichment

- Parametric enrichment $\rightsquigarrow \Lambda_{\ell+1} := \Lambda_\ell \cup \Upsilon_\ell$

  Key issue: allocation of meshes $\mathcal{T}_{(\ell+1)\mathbf{z}'}$ for each new collocation point $\mathbf{z}'$

Parametric enrichment

- Parametric enrichment $\rightsquigarrow \Lambda_{\ell+1} := \Lambda_\ell \cup \Upsilon_\ell$
  Key issue: allocation of meshes $\mathcal{T}_{(\ell+1)\mathbf{z}'}$ for each new collocation point $\mathbf{z}'$

- Set $\widetilde{\texttt{tol}} := (\#\mathcal{Y}_\ell)^{-1} \sum_{\mathbf{z}\in\mathcal{Y}_\ell} \mu_{\ell\mathbf{z}} \| L_{(\ell+1)\mathbf{z}} \|_{L^2_\pi(\Gamma)}$

- Parametric enrichment $\rightsquigarrow \Lambda_{\ell+1} := \Lambda_\ell \cup \Upsilon_\ell$

  Key issue: allocation of meshes $\mathcal{T}_{(\ell+1)\mathbf{z}'}$ for each new collocation point $\mathbf{z}'$

- Set $\widetilde{\mathtt{tol}} := (\#\mathcal{Y}_\ell)^{-1} \sum_{\mathbf{z}\in\mathcal{Y}_\ell} \mu_{\ell\mathbf{z}} \| L_{(\ell+1)\mathbf{z}} \|_{L^2_\pi(\Gamma)}$

- For each new collocation point $\mathbf{z}'$
  - Initialise the mesh $\mathcal{T}_{(\ell+1)\mathbf{z}'} := \mathcal{T}_0$
  - Iterate the standard adaptive loop

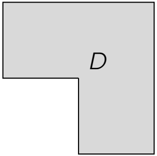
$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}$$

    until the resolution of the mesh $\mathcal{T}_{(\ell+1)\mathbf{z}'}$ is such that

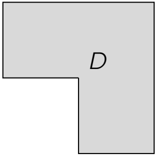
$$\mu_{(\ell+1)\mathbf{z}'} \| L_{(\ell+1)\mathbf{z}'} \|_{L^2_\pi(\Gamma)} < \widetilde{\mathtt{tol}}$$

Experiment 2: nonaffine parametric coefficient

- $-\nabla \cdot (a \nabla u) = 1$ in $D \times \Gamma$, $\quad u = 0$ on $\partial D \times \Gamma$
- $D := (-1, 1)^2 \setminus (-1, 0]^2 \rightsquigarrow$ L-shaped domain



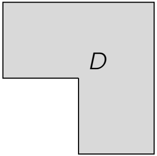
Experiment 2: nonaffine parametric coefficient



- $-\nabla \cdot (a \nabla u) = 1$ in $D \times \Gamma$, $\quad u = 0$ on $\partial D \times \Gamma$
- $D := (-1,1)^2 \setminus (-1,0]^2 \rightsquigarrow$ L-shaped domain
- Diffusion coefficient $\; a(x, \mathbf{y}) = exp(h(x, \mathbf{y}))$
  - $h(x, \mathbf{y}) = 1 + \sum_{m=1}^{M} \sqrt{\lambda_m}\, \varphi_m(x)\, y_m$
  - $\{(\lambda_m, \varphi_m(x))\}_{m=1}^{\infty}$ are the eigenpairs of $\displaystyle\int_{D \cup (-1,0]^2} \mathrm{Cov}[h](x, x')\, \varphi(x')\, \mathrm{d}x'$
  - $\mathrm{Cov}[h](x, x') = \sigma^2 \exp\left(-|x_1 - x_1'| - |x_2 - x_2'|\right)$
  - $\{y_m\}_{m \in \{1,\dots,M\}}$ are images of $U(-1,1)$ iid mean-zero r.v., $\mathrm{d}\pi_m(y_m) = \frac{1}{2}\, \mathrm{d}y_m$

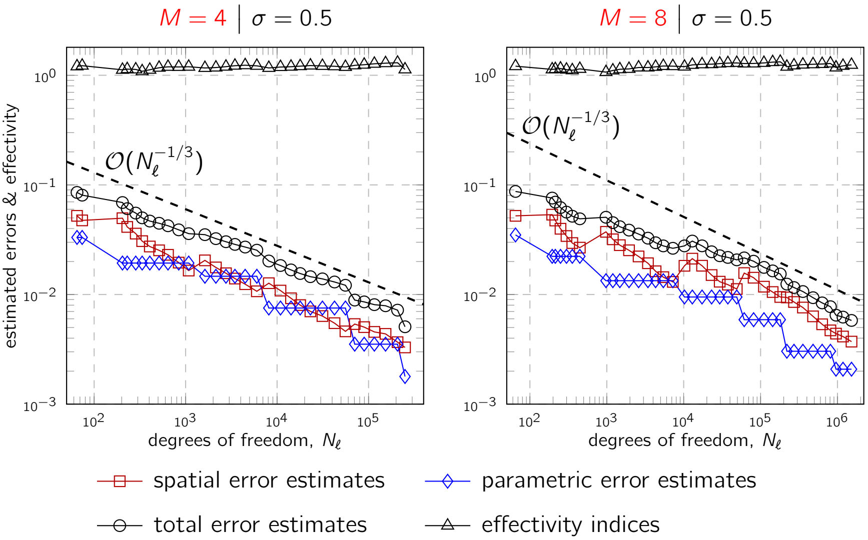
Experiment 2: nonaffine parametric coefficient

- $-\nabla \cdot (a \nabla u) = 1$ in $D \times \Gamma$, $\quad u = 0$ on $\partial D \times \Gamma$
- $D := (-1,1)^2 \setminus (-1,0]^2 \rightsquigarrow$ L-shaped domain
- Diffusion coefficient $\; a(x,\mathbf{y}) = exp(h(x,\mathbf{y}))$
  - $h(x,\mathbf{y}) = 1 + \sum_{m=1}^{M} \sqrt{\lambda_m}\, \varphi_m(x)\, y_m$
  - $\{(\lambda_m, \varphi_m(x))\}_{m=1}^{\infty}$ are the eigenpairs of $\displaystyle\int_{D \cup (-1,0]^2} \mathrm{Cov}[h](x,x')\, \varphi(x')\, \mathrm{d}x'$
  - $\mathrm{Cov}[h](x,x') = \sigma^2 \exp\left(-|x_1 - x_1'| - |x_2 - x_2'|\right)$
  - $\{y_m\}_{m \in \{1,\dots,M\}}$ are images of $U(-1,1)$ iid mean-zero r.v., $\mathrm{d}\pi_m(y_m) = \frac{1}{2}\,\mathrm{d}y_m$

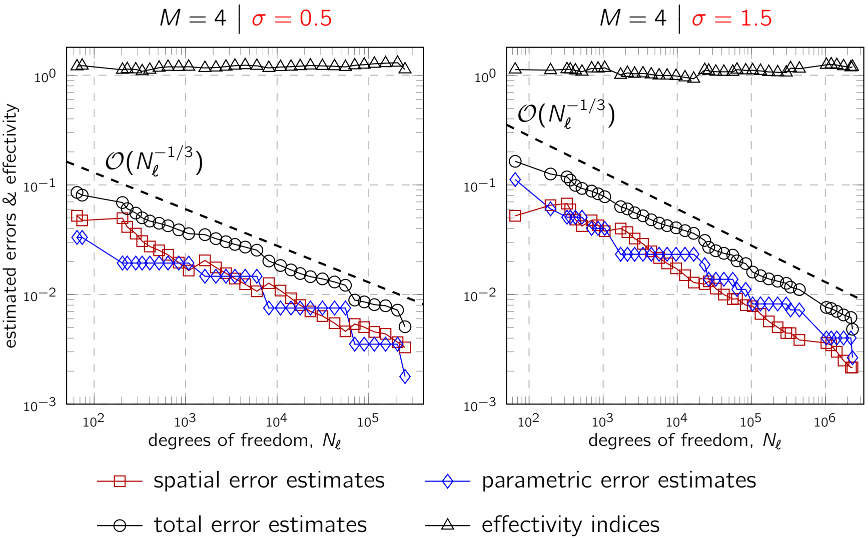


- Adaptive stochastic collocation FEM (Clenshaw–Curtis collocation points)

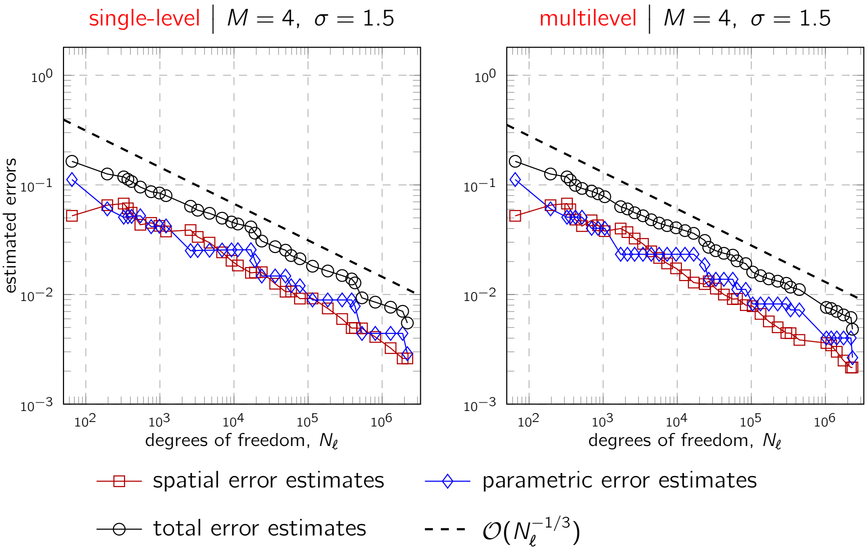
Experiment 2: effectivity and robustness of error estimates (1/2)



Experiment 2: effectivity and robustness of error estimates (2/2)



Experiment 2: single-level vs. multilevel refinement



Experiment 3: one peak problem

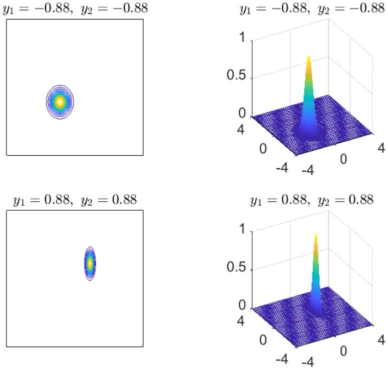
[Kornhuber, Youett '18], [Lang, Scheichl, Silvester '20]

- $-\nabla^2 u = f(x, \mathbf{y})$ in $D \times \Gamma$, $\quad u = g$ on $\partial D \times \Gamma$
- $D := (-4, 4)^2$, $\Gamma = [-1, 1]^2$, $\mathbf{y} = (y_1, y_2)$, $y_1, y_2 \sim U[-1, 1]$
- $u(x, \mathbf{y}) = \exp\big(-\frac{50}{16}\{\alpha(y_1)(x_1 - y_1)^2 + (x_2 - y_2)^2\}\big)$ with $\alpha(y_1) = (9y_1 + 11)/2$

Experiment 3: one peak problem

[Kornhuber, Youett '18], [Lang, Scheichl, Silvester '20]

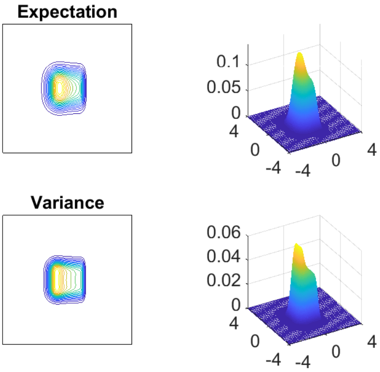
- $-\nabla^2 u = f(x, \mathbf{y})$ in $D \times \Gamma$, $\quad u = g$ on $\partial D \times \Gamma$
- $D := (-4, 4)^2$, $\Gamma = [-1, 1]^2$, $\mathbf{y} = (y_1, y_2)$, $y_1, y_2 \sim U[-1, 1]$
- $u(x, \mathbf{y}) = \exp\left(-\frac{50}{16}\{\alpha(y_1)(x_1 - y_1)^2 + (x_2 - y_2)^2\}\right)$ with $\alpha(y_1) = (9y_1 + 11)/2$



Experiment 3: one peak problem

[Kornhuber, Youett '18], [Lang, Scheichl, Silvester '20]

- $-\nabla^2 u = f(x, \mathbf{y})$ in $D \times \Gamma$, $\quad u = g$ on $\partial D \times \Gamma$
- $D := (-4, 4)^2$, $\Gamma = [-1, 1]^2$, $\mathbf{y} = (y_1, y_2)$, $y_1, y_2 \sim U[-1, 1]$
- $u(x, \mathbf{y}) = \exp\left(-\frac{50}{16}\{\alpha(y_1)(x_1 - y_1)^2 + (x_2 - y_2)^2\}\right)$ with $\alpha(y_1) = (9y_1 + 11)/2$



Experiment 3: one peak problem

[Kornhuber, Youett '18], [Lang, Scheichl, Silvester '20]

- $-\nabla^2 u = f(x, \mathbf{y})$ in $D \times \Gamma$, $\quad u = g$ on $\partial D \times \Gamma$
- $D := (-4, 4)^2$, $\Gamma = [-1, 1]^2$, $\mathbf{y} = (y_1, y_2)$, $y_1, y_2 \sim U[-1, 1]$
- $u(x, \mathbf{y}) = \exp\big(-\frac{50}{16}\{\alpha(y_1)(x_1 - y_1)^2 + (x_2 - y_2)^2\}\big)$ with $\alpha(y_1) = (9y_1 + 11)/2$
- Dirichlet b.c. for sampled FEM approximations: $u_{\bullet\mathbf{z}} = 0 \;\; \forall\, \mathbf{z} \in \mathcal{Y}_\bullet$
- Reference QoI

$$Q := \int_\Gamma \int_D \big(u(x, \mathbf{y})\big)^2 \,\mathrm{d}x \,\mathrm{d}\pi(\mathbf{y}) = 0.24152872\ldots$$

Experiment 3: one peak problem

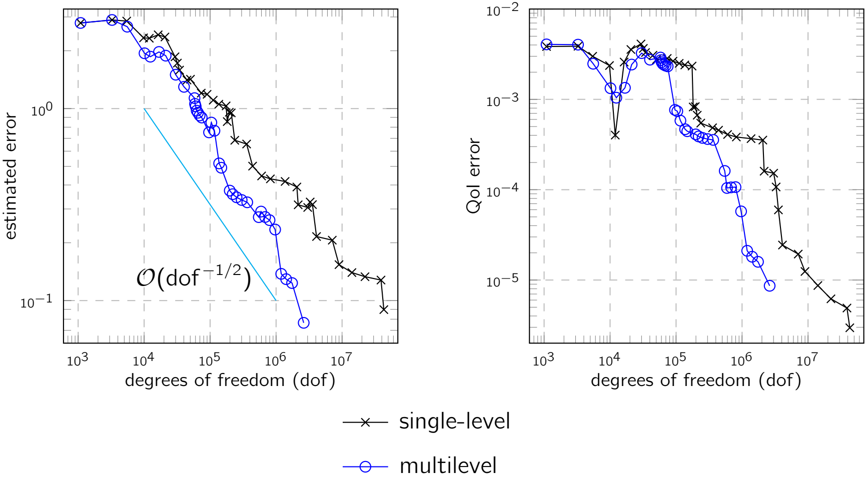
[Kornhuber, Youett '18], [Lang, Scheichl, Silvester '20]

- $-\nabla^2 u = f(x, \mathbf{y})$ in $D \times \Gamma$, $\quad u = g$ on $\partial D \times \Gamma$
- $D := (-4, 4)^2$, $\Gamma = [-1, 1]^2$, $\mathbf{y} = (y_1, y_2)$, $y_1, y_2 \sim U[-1, 1]$
- $u(x, \mathbf{y}) = \exp\left(-\frac{50}{16}\{\alpha(y_1)(x_1 - y_1)^2 + (x_2 - y_2)^2\}\right)$ with $\alpha(y_1) = (9y_1 + 11)/2$
- Dirichlet b.c. for sampled FEM approximations: $u_{\bullet \mathbf{z}} = 0 \ \ \forall\, \mathbf{z} \in \mathcal{Y}_\bullet$
- Reference QoI

$$Q := \int_\Gamma \int_D \big(u(x, \mathbf{y})\big)^2 \, \mathrm{d}x \, \mathrm{d}\pi(\mathbf{y}) = 0.24152872\ldots$$

- Adaptive stochastic collocation FEM (Clenshaw–Curtis collocation points)
  - single-level *vs.* multilevel refinement

Experiment 3: single-level vs. multilevel refinement (1/2)

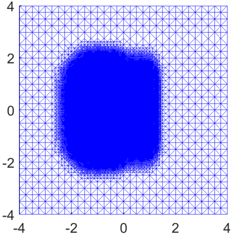


Experiment 3: single-level vs. multilevel refinement (2/2)

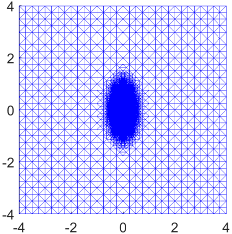
	single-level SC-FEM	multilevel SC-FEM
# iterations	38	34
# collocation points	169	153
final #dof	42'961'659	2'620'343
$\lvert Q(u) - Q(u^{\mathrm{SC}}) \rvert$	4.736e−5	1.381e−4

Experiment 3: single-level vs. multilevel refinement (2/2)

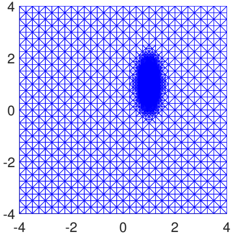
	single-level SC-FEM	multilevel SC-FEM
# iterations	38	34
# collocation points	169	153
final #dof	42’961’659	2’620’343
$\lvert Q(u) - Q(u^{\mathrm{SC}}) \rvert$	4.736e−5	1.381e−4



single-level
mesh $\mathcal{T}_\ell$



mesh $\mathcal{T}_{\ell \mathbf{z}}$
$\mathbf{z} = (0, 0)$



mesh $\mathcal{T}_{\ell \mathbf{z}}$
$\mathbf{z} = (1, 1)$

Conclusions, extensions and outlook

- For multilevel stochastic Galerkin FEM
  - exploiting Galerkin orthogonality & properties of orthogonal polynomials
  - novel reliable and efficient a posteriori error estimates
  - rate optimal adaptive algorithms for PDEs with *affine-parametric* inputs
  - extensions: goal-oriented adaptivity; parameter-dependent linear elasticity

Conclusions, extensions and outlook

- For multilevel stochastic Galerkin FEM
  - exploiting Galerkin orthogonality & properties of orthogonal polynomials
  - novel reliable and efficient a posteriori error estimates
  - rate optimal adaptive algorithms for PDEs with *affine-parametric* inputs
  - extensions: goal-oriented adaptivity; parameter-dependent linear elasticity
- For multilevel stochastic collocation FEM
  - reliable, effective and robust a posteriori error estimation strategy
  - applicable to problems with *affine and nonaffine* parametric inputs
  - practical error indicators for multilevel adaptivity
  - optimal convergence rates do not seem to be feasible in general
  - optimal rates can be recovered for problems with parameter-dependent local spatial features
  - outlook: convergence analysis, goal-oriented adaptivity, ...

References & Software

- Key references

  - AB, D. Silvester, *Efficient adaptive stochastic Galerkin methods for parametric operator equations*, SIAM J. Sci. Comp., Vol. 38 (2016), pp. A2118–A2140.
  - AB, D. Praetorius, L. Rocchi, M. Ruggeri, *Convergence of adaptive stochastic Galerkin FEM*, SIAM J. Numer. Anal., Vol. 57 (2019), pp. 2359–2382.
  - AB, D. Praetorius, M. Ruggeri, *Two-level a posteriori error estimation for adaptive multilevel stochastic Galerkin FEM*, SIAM/ASA J. Uncertain. Quantif., Vol. 9 (2021), pp. 1184–1216.
  - AB, D. Praetorius, M. Ruggeri, *Convergence and rate optimality of adaptive multilevel stochastic Galerkin FEM*, IMA J. Numer. Anal., Vol. 42 (2022), pp. 2190–2213.
  - AB, D. Silvester, F. Xu, *Error estimation and adaptivity for stochastic collocation finite elements. Part I: single-level approximation*, SIAM J. Sci. Comp., Vol. 44 (2022), pp. A3393–A3412.
  - AB, D. Silvester, *Error estimation and adaptivity for stochastic collocation finite elements. Part II: multilevel approximation*, SIAM J. Sci. Comp. (to appear).

- Software
  - Stochastic T-IFISS, `https://github.com/albespalov/Stochastic_T-IFISS`
  - Adaptive ML-SCFEM, `https://github.com/albespalov/Adaptive_ML-SCFEM`