Hybrid high-order methods for the biharmonic problem

Alexandre Ern

ENPC and INRIA, Paris, France
joint work with Zhaonan Dong (INRIA)

Irish Numerical Analysis Forum, 12/05/2022

Outline

- HHO for Poisson model problem
- HHO for biharmonic problem
- Numerical results
- Error analysis with low regularity

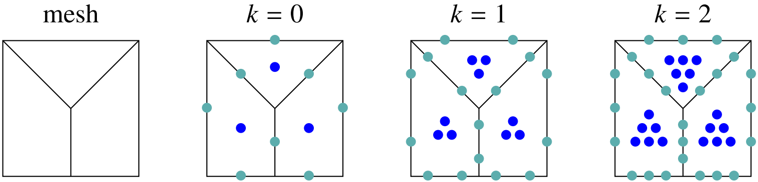
HHO for Poisson model problem

HHO methods: basic ideas

- Introduced in [Di Pietro, AE, Lemaire 14] (linear diffusion) and [Di Pietro, AE 15] (locking-free linear elasticity)
- Degrees of freedom (dofs) attached to mesh cells and faces

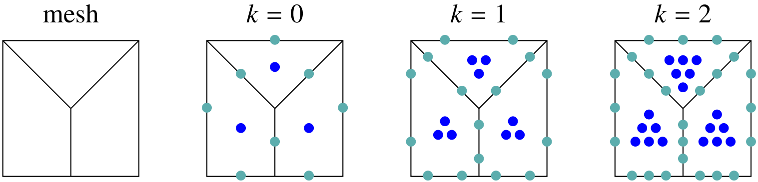
HHO methods: basic ideas

- Introduced in [Di Pietro, AE, Lemaire 14] (linear diffusion) and [Di Pietro, AE 15] (locking-free linear elasticity)
- Degrees of freedom (dofs) attached to mesh cells and faces
- Let us start with polynomials of the same degree $k \geq 0$ on cells and faces (dots do not mean point evaluation here)



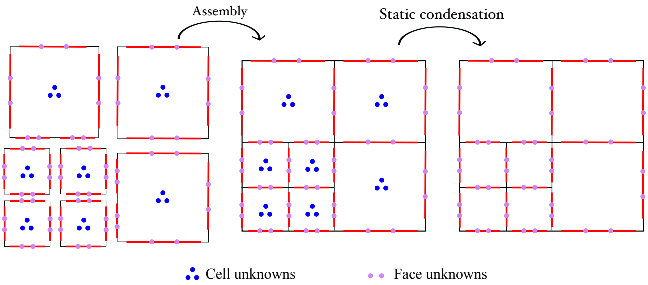
HHO methods: basic ideas

- Introduced in [Di Pietro, AE, Lemaire 14] (linear diffusion) and [Di Pietro, AE 15] (locking-free linear elasticity)
- Degrees of freedom (dofs) attached to mesh cells and faces
- Let us start with polynomials of the same degree $k \geq 0$ on cells and faces (dots do not mean point evaluation here)



- In each cell, one devises a local gradient reconstruction operator
- One adds local stabilization to weakly enforce the matching of cell dof traces with face dofs

Assembly and static condensation



- Global dofs $\hat{u}_h = (u_{\mathcal{T}}, u_{\mathcal{F}})$ ($\mathcal{T} :=$ {mesh cells}, $\mathcal{F} :=$ {mesh faces})

$$\hat{U}_h := \mathbb{P}^k(\mathcal{T}) \times \mathbb{P}^k(\mathcal{F}), \quad \mathbb{P}^k(\mathcal{T}) := \bigtimes_{T \in \mathcal{T}} \mathbb{P}^k(T), \quad \mathbb{P}^k(\mathcal{F}) := \bigtimes_{F \in \mathcal{F}} \mathbb{P}^k(F)$$

- Cell dofs eliminated locally by static condensation
  - only face dofs are globally coupled
  - cell dofs recovered by local post-processing

- Dirichlet conditions enforced on face boundary dofs $\longrightarrow$ subspace $\hat{U}_{h0}$

Main assets of HHO methods

- General meshes: polytopal cells, hanging nodes
- Optimal error estimates
  - $O(h^t)$ H^1-error estimate if $u \in H^{1+t}(\Omega)$, $t \in (\frac{1}{2}, k+1]$

    face dofs of order $k \geq 0 \implies O(h^{k+1})$ H^1-error estimate

  - duality argument for L^2-error estimate

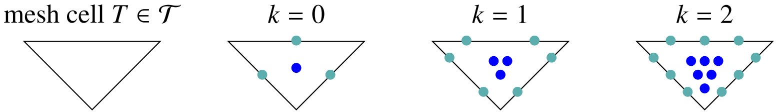
Main assets of HHO methods

- General meshes: polytopal cells, hanging nodes
- Optimal error estimates
  - $O(h^t)$ H^1-error estimate if $u \in H^{1+t}(\Omega)$, $t \in (\frac{1}{2}, k+1]$

    face dofs of order $k \geq 0 \implies O(h^{k+1})$ H^1-error estimate

  - duality argument for L^2-error estimate
- Local conservation
  - optimally convergent and algebraically balanced fluxes on faces
  - as any face-based method, balance at cell level
- Attractive computational costs
  - only face dofs are globally coupled
  - compact stencil

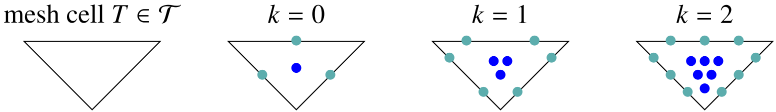
Local dofs and gradient reconstruction



- $\hat{u}_T = (u_T, u_{\partial T})$ with cell dofs $u_T \in \mathbb{P}^k(T)$ and face dofs $u_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T \in \hat{U}_T := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}), \qquad \mathbb{P}^k(\mathcal{F}_{\partial T}) := \bigtimes_{F \in \mathcal{F}_{\partial T}} \mathbb{P}^k(F)$$

Local dofs and gradient reconstruction



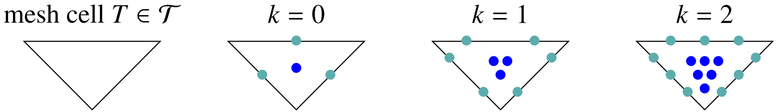
- $\hat{u}_T = (u_T, u_{\partial T})$ with cell dofs $u_T \in \mathbb{P}^k(T)$ and face dofs $u_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T \in \hat{U}_T := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}), \qquad \mathbb{P}^k(\mathcal{F}_{\partial T}) := \bigtimes_{F \in \mathcal{F}_{\partial T}} \mathbb{P}^k(F)$$

- Potential reconstruction $R_T : \hat{U}_T \longrightarrow \mathbb{P}^{k+1}(T)$
- Main idea: mimic integration by parts (smooth functions u, q):

$$(\nabla u, \nabla q)_T = -(u, \Delta q)_T + (u, \nabla q \cdot \mathbf{n}_T)_{\partial T}$$

Local dofs and gradient reconstruction



- $\hat{u}_T = (u_T, u_{\partial T})$ with cell dofs $u_T \in \mathbb{P}^k(T)$ and face dofs $u_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})$

$$\hat{u}_T \in \hat{U}_T := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}), \qquad \mathbb{P}^k(\mathcal{F}_{\partial T}) := \bigtimes_{F \in \mathcal{F}_{\partial T}} \mathbb{P}^k(F)$$

- Potential reconstruction $R_T : \hat{U}_T \longrightarrow \mathbb{P}^{k+1}(T)$
- Main idea: mimic integration by parts (smooth functions u, q):

$$(\nabla u, \nabla q)_T = -(u, \Delta q)_T + (u, \nabla q \cdot \mathbf{n}_T)_{\partial T}$$

- We require that $\forall q \in \mathbb{P}^{k+1}(T)/\mathbb{P}^0$,

$$(\nabla R_T(\hat{u}_T), \nabla q)_T = -(u_T, \Delta q)_T + (u_{\partial T}, \nabla q \cdot \mathbf{n}_T)_{\partial T}$$

  together with $(R_T(\hat{u}_T), 1)_T = (u_T, 1)_T$
- Gradient reconstruction $\mathbf{G}_T(\hat{u}_T) := \nabla R_T(\hat{u}_T) \in [\mathbb{P}^k(T)]^d$

Local stabilization and bilinear form

- In all cases, the local bilinear form writes

$$a_T(\hat{u}_T, \hat{w}_T) := \underbrace{(\nabla R_T(\hat{u}_T), \nabla R_T(\hat{w}_T))_T}_{\approx (\nabla u, \nabla w)_T} + \underbrace{h_T^{-1}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T}}_{\text{weakly enforces } u_T|_{\partial T} - u_{\partial T} \approx 0}$$

Local stabilization and bilinear form

- In all cases, the local bilinear form writes

$$a_T(\hat{u}_T, \hat{w}_T) := \underbrace{(\nabla R_T(\hat{u}_T), \nabla R_T(\hat{w}_T))_T}_{\approx (\nabla u, \nabla w)_T} + \underbrace{h_T^{-1}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T}}_{\text{weakly enforces } u_T|_{\partial T} - u_{\partial T} \approx 0}$$

- Local stabilization operator acting on $\delta := u_T|_{\partial T} - u_{\partial T}$

$$S_{\partial T}(\hat{u}_T) := \Pi_{\partial T}^k \Big(\delta - \underbrace{\big((I - \Pi_T^k) R_T(0, \delta)\big)|_{\partial T}}_{\text{high-order correction}} \Big)$$

Local stabilization and bilinear form

- In all cases, the local bilinear form writes

$$a_T(\hat{u}_T, \hat{w}_T) := \underbrace{(\nabla R_T(\hat{u}_T), \nabla R_T(\hat{w}_T))_T}_{\approx(\nabla u, \nabla w)_T} + \underbrace{h_T^{-1}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T}}_{\text{weakly enforces } u_T|_{\partial T} - u_{\partial T} \approx 0}$$

- Local stabilization operator acting on $\delta := u_T|_{\partial T} - u_{\partial T}$

$$S_{\partial T}(\hat{u}_T) := \Pi_{\partial T}^k\Big(\delta - \underbrace{((I - \Pi_T^k)R_T(0, \delta))|_{\partial T}}_{\text{high-order correction}}\Big)$$

- (Important) variant on cell dofs and stabilization
  - mixed-order setting: $(k+1)$ for cell dofs and $k \geq 0$ for face dofs
  - Lehrenfeld–Schöberl HDG stabilization

$$S_{\partial T}(\hat{u}_T) := \Pi_{\partial T}^k(\delta)$$

  - slightly higher cost for static condensation compensated by lower cost for computing stabilization

Link to other methods

- HHO($k = 0$) equivalent (up to stab.) to Hybrid FV and Hybrid Mimetic Mixed methods [Eymard, Gallouet, Herbin 10; Droniou et al. 10]

Link to other methods

- HHO($k = 0$) equivalent (up to stab.) to Hybrid FV and Hybrid Mimetic Mixed methods [Eymard, Gallouet, Herbin 10; Droniou et al. 10]
- HHO fits into HDG setting [Cockburn, Di Pietro, AE 16]
  - flux variable in HDG ↔ HHO grad. rec.
  - numerical flux trace in HHO is $-\nabla R_T(\hat{u}_T)\cdot\mathbf{n}_T + h_T^{-1}(\tilde{S}_{\partial T}^{*} \circ \tilde{S}_{\partial T})(\delta)$
  - HHO allows for a simpler analysis based on L^2-projections: avoids special HDG projection

Link to other methods

- HHO($k = 0$) equivalent (up to stab.) to Hybrid FV and Hybrid Mimetic Mixed methods [Eymard, Gallouet, Herbin 10; Droniou et al. 10]
- HHO fits into HDG setting [Cockburn, Di Pietro, AE 16]
  - flux variable in HDG ↔ HHO grad. rec.
  - numerical flux trace in HHO is $-\nabla R_T(\hat{u}_T)\cdot\mathbf{n}_T + h_T^{-1}(\tilde{S}^*_{\partial T} \circ \tilde{S}_{\partial T})(\delta)$
  - HHO allows for a simpler analysis based on L^2-projections: avoids special HDG projection
- Similar devising of HHO and weak Galerkin methods [Wang, Ye 13]
  - weak gradient ↔ HHO grad. rec.
  - WG often uses plain LS stabilization (in general, suboptimal: face dofs of order $k \geq 0 \implies O(h^k)$ H^1-estimate)

Link to other methods

- HHO($k = 0$) equivalent (up to stab.) to Hybrid FV and Hybrid Mimetic Mixed methods [Eymard, Gallouet, Herbin 10; Droniou et al. 10]

- HHO fits into HDG setting [Cockburn, Di Pietro, AE 16]
  - flux variable in HDG $\leftrightarrow$ HHO grad. rec.
  - numerical flux trace in HHO is $-\nabla R_T(\hat{u}_T)\cdot\mathbf{n}_T + h_T^{-1}(\tilde{S}^*_{\partial T} \circ \tilde{S}_{\partial T})(\delta)$
  - HHO allows for a simpler analysis based on L^2-projections: avoids special HDG projection

- Similar devising of HHO and weak Galerkin methods [Wang, Ye 13]
  - weak gradient $\leftrightarrow$ HHO grad. rec.
  - WG often uses plain LS stabilization (in general, suboptimal: face dofs of order $k \geq 0 \implies O(h^k)$ H^1-estimate)

- HHO equivalent (up to stab.) to ncVEM [Ayuso, Manzini, Lipnikov 16]
  - HHO dof space $\hat{U}_T$ isomorphic to virtual space $\mathcal{V}_T$

$$\mathbb{P}^{k+1}(T) \subsetneq \mathcal{V}_T := \{v \in H^1(T) \mid \Delta v \in \mathbb{P}^k(T),\ \mathbf{n}\cdot\nabla v|_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})\}$$

  - see [Chaumont, AE, Lemaire, Valentin 21] for equivalence with MHM

Link to other methods

- HHO($k = 0$) equivalent (up to stab.) to Hybrid FV and Hybrid Mimetic Mixed methods [Eymard, Gallouet, Herbin 10; Droniou et al. 10]

- HHO fits into HDG setting [Cockburn, Di Pietro, AE 16]
  - flux variable in HDG $\leftrightarrow$ HHO grad. rec.
  - numerical flux trace in HHO is $-\nabla R_T(\hat{u}_T)\cdot\mathbf{n}_T + h_T^{-1}(\tilde{S}^*_{\partial T} \circ \tilde{S}_{\partial T})(\delta)$
  - HHO allows for a simpler analysis based on L^2-projections: avoids special HDG projection

- Similar devising of HHO and weak Galerkin methods [Wang, Ye 13]
  - weak gradient $\leftrightarrow$ HHO grad. rec.
  - WG often uses plain LS stabilization (in general, suboptimal: face dofs of order $k \geq 0 \implies O(h^k)\ H^1$-estimate)

- HHO equivalent (up to stab.) to ncVEM [Ayuso, Manzini, Lipnikov 16]
  - HHO dof space $\hat{U}_T$ isomorphic to virtual space $\mathcal{V}_T$

$$\mathbb{P}^{k+1}(T) \subsetneq \mathcal{V}_T := \{v \in H^1(T) \mid \Delta v \in \mathbb{P}^k(T),\ \mathbf{n}\cdot\nabla v|_{\partial T} \in \mathbb{P}^k(\mathcal{F}_{\partial T})\}$$

  - see [Chaumont, AE, Lemaire, Valentin 21] for equivalence with MHM

- Different devising viewpoints should be mutually enriching!

- Broad area of applications (non-exhaustive list...)
  - **solid mechanics**: nonlinear elasticity, hyperlasticity and plasticity, contact, Tresca friction, obstacle pb
  - **fluid mechanics/porous media**: Stokes, NS, poroelasticity, fractures
  - Leray-Lions, spectral pb, H^{-1}-loads, magnetostatics, de Rham complexes

Applications, libraries, textbooks

- Broad area of applications (non-exhaustive list...)
  - **solid mechanics**: nonlinear elasticity, hyperlasticity and plasticity, contact, Tresca friction, obstacle pb
  - **fluid mechanics/porous media**: Stokes, NS, poroelasticity, fractures
  - Leray-Lions, spectral pb, H^{-1}-loads, magnetostatics, de Rham complexes

- Libraries
  - industry (`code_aster`, `code_saturne`, EDF R&D), ongoing developments at CEA
  - academia: `diskpp (C++)` (ENPC/INRIA `github.com/wareHHOuse`), HArD::Core (Monash/Montpellier `github.com/jdroniou/HArDCore`)

Applications, libraries, textbooks

- Broad area of applications (non-exhaustive list...)
  - **solid mechanics**: nonlinear elasticity, hyperlasticity and plasticity, contact, Tresca friction, obstacle pb
  - **fluid mechanics/porous media**: Stokes, NS, poroelasticity, fractures
  - Leray-Lions, spectral pb, H^{-1}-loads, magnetostatics, de Rham complexes
- Libraries
  - industry (`code_aster`, `code_saturne`, EDF R&D), ongoing developments at CEA
  - academia: `diskpp (C++)` (ENPC/INRIA `github.com/wareHHOuse`), HArD::Core (Monash/Montpellier `github.com/jdroniou/HArDCore`)
- Textbooks
  - Di Pietro, Droniou, The HHO method for polytopal meshes. Design, analysis and applications (Springer, 2020)
  - Cicuttin, AE, Pignet, HHO methods. A primer with application to solid mechanics (Springer Briefs, 2021)

Main ideas in error analysis (1/3)

- Recall $a_T(\hat{u}_T, \hat{w}_T) := (\nabla R_T(\hat{u}_T), \nabla R_T(\hat{w}_T))_T + h_T^{-1}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T}$
- Discrete problem: Find $\hat{u}_h \in \hat{U}_{h0}$ s.t.

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) = (f, w_{\mathcal{T}})_\Omega, \qquad \forall \hat{w}_h \in \hat{U}_{h0}$$

Main ideas in error analysis (1/3)

- Recall $a_T(\hat{u}_T, \hat{w}_T) := (\nabla R_T(\hat{u}_T), \nabla R_T(\hat{w}_T))_T + h_T^{-1}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T}$
- Discrete problem: Find $\hat{u}_h \in \hat{U}_{h0}$ s.t.

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) = (f, w_{\mathcal{T}})_\Omega, \qquad \forall \hat{w}_h \in \hat{U}_{h0}$$

- Stability and boundedness: There are $0 < \alpha \leq \omega$ s.t. for all $T \in \mathcal{T}$,

$$\alpha \|\hat{u}_T\|_{\hat{U}_T}^2 \leq a_T(\hat{u}_T, \hat{u}_T) \leq \omega \|\hat{u}_T\|_{\hat{U}_T}^2, \quad \forall \hat{u}_T \in \hat{U}_T$$

with $\|\hat{u}_T\|_{\hat{U}_T}^2 := \|\nabla u_T\|_T^2 + h_T^{-1}\|u_T|_{\partial T} - u_{\partial T}\|_{\partial T}^2$

Main ideas in error analysis (1/3)

- Recall $a_T(\hat{u}_T, \hat{w}_T) := (\nabla R_T(\hat{u}_T), \nabla R_T(\hat{w}_T))_T + h_T^{-1}(S_{\partial T}(\hat{u}_T), S_{\partial T}(\hat{w}_T))_{\partial T}$
- Discrete problem: Find $\hat{u}_h \in \hat{U}_{h0}$ s.t.

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) = (f, w_{\mathcal{T}})_\Omega, \qquad \forall \hat{w}_h \in \hat{U}_{h0}$$

- Stability and boundedness: There are $0 < \alpha \leq \omega$ s.t. for all $T \in \mathcal{T}$,

$$\alpha \|\hat{u}_T\|^2_{\hat{U}_T} \leq a_T(\hat{u}_T, \hat{u}_T) \leq \omega \|\hat{u}_T\|^2_{\hat{U}_T}, \quad \forall \hat{u}_T \in \hat{U}_T$$

  with $\|\hat{u}_T\|^2_{\hat{U}_T} := \|\nabla u_T\|^2_T + h_T^{-1}\|u_T|_{\partial T} - u_{\partial T}\|^2_{\partial T}$
- $\|\hat{u}_h\|^2_{\hat{U}_h} := \sum_{T \in \mathcal{T}} \|\hat{u}_T\|^2_{\hat{U}_T}$ defines a norm on $\hat{U}_{h0}$
- Discrete problem is well-posed (Lax–Milgram lemma)

Main ideas in error analysis (2/3)

- Local approximation operator $J_T^{\mathrm{HHO}} : H^1(T) \rightarrow \mathbb{P}^{k+1}(T)$

$$J_T^{\mathrm{HHO}} : H^1(T) \xrightarrow{\hat{I}_T} \hat{U}_T \xrightarrow{R_T} \mathbb{P}^{k+1}(T), \qquad \hat{I}_T(v) := (\Pi_T^k(v), \Pi_{\partial T}^k(v|_{\partial T}))$$

  - J_T^{HHO} is the elliptic projector onto $\mathbb{P}^{k+1}(T)$
  - $h_T^{-\frac{1}{2}} \|S_{\partial T}(\hat{I}_T(v))\|_{\partial T} \lesssim \|\nabla(v - J_T^{\mathrm{HHO}}(v))\|_T \lesssim h_T^{k+1} |v|_{H^{k+2}(T)}$

Main ideas in error analysis (2/3)

- Local approximation operator $J_T^{\text{HHO}} : H^1(T) \to \mathbb{P}^{k+1}(T)$

$$J_T^{\text{HHO}} : H^1(T) \xrightarrow{\hat{I}_T} \hat{U}_T \xrightarrow{R_T} \mathbb{P}^{k+1}(T), \qquad \hat{I}_T(v) := (\Pi_T^k(v), \Pi_{\partial T}^k(v|_{\partial T}))$$

  - J_T^{HHO} is the elliptic projector onto $\mathbb{P}^{k+1}(T)$
  - $h_T^{-\frac{1}{2}} \|S_{\partial T}(\hat{I}_T(v))\|_{\partial T} \lesssim \|\nabla(v - J_T^{\text{HHO}}(v))\|_T \lesssim h_T^{k+1}|v|_{H^{k+2}(T)}$

- Assume exact solution u is in $H^{1+s}(\Omega)$, $s > \frac{1}{2}$
- Set $\|v\|_{\sharp,T}^2 := \|\nabla v\|_T^2 + h_T\|\nabla v{\cdot}\mathbf{n}_T\|_{\partial T}^2$ and $\|v\|_{\sharp,\mathcal{T}}^2 := \sum_{T\in\mathcal{T}} \|v\|_{\sharp,T}^2$
- The following error estimate holds:

$$\|\nabla_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_\Omega \lesssim \|u - J_{\mathcal{T}}^{\text{HHO}}(u)\|_{\sharp,\mathcal{T}}$$

  with $R_{\mathcal{T}}$ and $J_{\mathcal{T}}^{\text{HHO}}$ defined cellwise using R_T and J_T^{HHO}

Main ideas in error analysis (2/3)

- Local approximation operator $J_T^{\text{HHO}} : H^1(T) \to \mathbb{P}^{k+1}(T)$

$$J_T^{\text{HHO}} : H^1(T) \xrightarrow{\hat{I}_T} \hat{U}_T \xrightarrow{R_T} \mathbb{P}^{k+1}(T), \qquad \hat{I}_T(v) := (\Pi_T^k(v), \Pi_{\partial T}^k(v|_{\partial T}))$$

  - J_T^{HHO} is the elliptic projector onto $\mathbb{P}^{k+1}(T)$
  - $h_T^{-\frac{1}{2}} \|S_{\partial T}(\hat{I}_T(v))\|_{\partial T} \lesssim \|\nabla(v - J_T^{\text{HHO}}(v))\|_T \lesssim h_T^{k+1}|v|_{H^{k+2}(T)}$

- Assume exact solution u is in $H^{1+s}(\Omega)$, $s > \frac{1}{2}$

- Set $\|v\|_{\sharp,T}^2 := \|\nabla v\|_T^2 + h_T\|\nabla v{\cdot}\mathbf{n}_T\|_{\partial T}^2$ and $\|v\|_{\sharp,\mathcal{T}}^2 := \sum_{T\in\mathcal{T}} \|v\|_{\sharp,T}^2$

- The following error estimate holds:

$$\|\nabla_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_\Omega \lesssim \|u - J_{\mathcal{T}}^{\text{HHO}}(u)\|_{\sharp,\mathcal{T}}$$

  with $R_{\mathcal{T}}$ and $J_{\mathcal{T}}^{\text{HHO}}$ defined cellwise using R_T and J_T^{HHO}

- If $u \in H^{1+t}(\Omega)$ with $t \in (\frac{1}{2}, k+1]$, $\|\nabla_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_\Omega \lesssim h^t|u|_{H^{1+t}(\Omega)}$

Main ideas in error analysis (3/3)

- Bound on consistency error: For all $\hat{w}_h \in \hat{U}_{h0}$,

$$(f, w_{\mathcal{T}})_\Omega = \sum_{T\in\mathcal{T}} (-\Delta u, w_T)_T = \sum_{T\in\mathcal{T}} (\nabla u, \nabla w_T)_T - (\nabla u{\cdot}\mathbf{n}_T, w_T)_{\partial T}$$
$$= \sum_{T\in\mathcal{T}} (\nabla u, \nabla w_T)_T - (\nabla u{\cdot}\mathbf{n}_T, w_T - w_{\partial T})_{\partial T}$$

Key step where regularity assumption $u \in H^{1+s}(\Omega)$, $s > \frac{1}{2}$, is used

Main ideas in error analysis (3/3)

- Bound on consistency error: For all $\hat{w}_h \in \hat{U}_{h0}$,

$$(f, w_\mathcal{T})_\Omega = \sum_{T\in\mathcal{T}} (-\Delta u, w_T)_T = \sum_{T\in\mathcal{T}} (\nabla u, \nabla w_T)_T - (\nabla u\cdot \mathbf{n}_T, w_T)_{\partial T}$$
$$= \sum_{T\in\mathcal{T}} (\nabla u, \nabla w_T)_T - (\nabla u\cdot \mathbf{n}_T, w_T - w_{\partial T})_{\partial T}$$

  Key step where regularity assumption $u \in H^{1+s}(\Omega)$, $s > \frac{1}{2}$, is used

- Recalling $J_T^{\mathrm{HHO}} = R_T \circ \hat{I}_T$ and definition of $R_T(\hat{w}_T)$ gives

$$\chi(\hat{w}_h) := (f, w_\mathcal{T})_\Omega - \sum_{T\in\mathcal{T}} a_T(\hat{I}_T(u), \hat{w}_T) = (f, w_\mathcal{T})_\Omega - \sum_{T\in\mathcal{T}} (\nabla J_T^{\mathrm{HHO}}(u), \nabla R_T(\hat{w}_T))_T + \text{stb.}$$
$$= \sum_{T\in\mathcal{T}} (\nabla \eta, \nabla w_T)_T - (\nabla \eta\cdot \mathbf{n}_T, w_T - w_{\partial T})_{\partial T} + \text{stb.}$$

  with $\eta|_T := u|_T - J_T^{\mathrm{HHO}}(u)$, ... so that $|\chi(\hat{w}_h)| \lesssim \|\eta\|_{\sharp,\mathcal{T}} \|\hat{w}_h\|_{\hat{U}_h}$

- Bound on consistency error: For all $\hat{w}_h \in \hat{U}_{h0}$,

$$(f, w_{\mathcal{T}})_\Omega = \sum_{T\in\mathcal{T}} (-\Delta u, w_T)_T = \sum_{T\in\mathcal{T}} (\nabla u, \nabla w_T)_T - (\nabla u\cdot\mathbf{n}_T, w_T)_{\partial T}$$
$$= \sum_{T\in\mathcal{T}} (\nabla u, \nabla w_T)_T - (\nabla u\cdot\mathbf{n}_T, w_T - w_{\partial T})_{\partial T}$$

Key step where regularity assumption $u \in H^{1+s}(\Omega)$, $s > \frac{1}{2}$, is used

- Recalling $J_T^{\mathrm{HHO}} = R_T \circ \hat{I}_T$ and definition of $R_T(\hat{w}_T)$ gives

$$\chi(\hat{w}_h) := (f, w_{\mathcal{T}})_\Omega - \sum_{T\in\mathcal{T}} a_T(\hat{I}_T(u), \hat{w}_T) = (f, w_{\mathcal{T}})_\Omega - \sum_{T\in\mathcal{T}} (\nabla J_T^{\mathrm{HHO}}(u), \nabla R_T(\hat{w}_T))_T + \text{stb.}$$
$$= \sum_{T\in\mathcal{T}} (\nabla \eta, \nabla w_T)_T - (\nabla \eta\cdot\mathbf{n}_T, w_T - w_{\partial T})_{\partial T} + \text{stb.}$$

with $\eta|_T := u|_T - J_T^{\mathrm{HHO}}(u)$, ... so that $|\chi(\hat{w}_h)| \lesssim \|\eta\|_{\sharp,\mathcal{T}} \|\hat{w}_h\|_{\hat{U}_h}$

- Regularity assumption $s > \frac{1}{2}$ is classical for any nonconforming method (CR, Nitsche, dG, HDG, ...); how to circumvent it?
  - modify RHS using suitable bubble functions; see [Veeser, Zanotti, 18-] for general theory and [AE, Zanotti, 20] for HHO $\Longrightarrow$ optimal in H^1
  - keep RHS but give weaker meaning to facewise normal derivative [AE, Guermond 21 (FoCM)] $\Longrightarrow$ allow for any $s > 0$

HHO for biharmonic problem

Model problem

- Open, bounded, polytopal Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$
- Load $f \in L^2(\Omega)$

$$\Delta^2 u = f \quad + \quad \text{BC's} \begin{cases} u = 0,\ \partial_n u = 0 & \text{(type I)} \\ u = 0,\ \partial_{nn} u = 0 & \text{(type II)} \end{cases}$$

Model problem

- Open, bounded, polytopal Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$
- Load $f \in L^2(\Omega)$

$$\Delta^2 u = f \quad + \quad \text{BC's} \begin{cases} u = 0,\ \partial_n u = 0 & \text{(type I)} \\ u = 0,\ \partial_{nn} u = 0 & \text{(type II)} \end{cases}$$

- Focusing on type I BC's, the weak formulation is

$$\text{Find } u \in H_0^2(\Omega) \text{ s.t.} \quad (\nabla^2 u, \nabla^2 w)_\Omega = (f, w)_\Omega \quad \forall w \in H_0^2(\Omega)$$

  This problem is well-posed (Lax–Milgram lemma)

- It is also possible to consider type II BC's, non-homogeneous BC's, and mix both BC's

Local HHO dofs

- Recall for second-order PDEs that local HHO dofs comprise
  - cell dofs to approximate the solution in mesh cells
  - face dofs to approximate the solution trace on mesh faces

$$\hat{U}_T := \mathbb{P}^{k+1}(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \quad \text{or} \quad \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \quad k \geq 0$$

Local HHO dofs

- Recall for second-order PDEs that local HHO dofs comprise
  - cell dofs to approximate the solution in mesh cells
  - face dofs to approximate the solution trace on mesh faces

$$\hat{U}_T := \mathbb{P}^{k+1}(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \quad \text{or} \quad \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \quad k \geq 0$$

- For biharmonic problem, we need additional face dofs
  - either approximating the full gradient trace on mesh faces (vector-valued)
  - or just the normal derivative on mesh faces (scalar-valued)

Local HHO dofs

- Recall for second-order PDEs that local HHO dofs comprise
  - cell dofs to approximate the solution in mesh cells
  - face dofs to approximate the solution trace on mesh faces

$$\hat{U}_T := \mathbb{P}^{k+1}(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \quad \text{or} \quad \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \quad k \geq 0$$

- For biharmonic problem, we need additional face dofs
  - either approximating the full gradient trace on mesh faces (vector-valued)
  - or just the normal derivative on mesh faces (scalar-valued)

- The choice studied in [Bonaldi, Di Pietro, Geymonat, Krasucki, 18] is

$$\hat{U}_T := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \times [\mathbb{P}^k(\mathcal{F}_{\partial T})]^d \quad k \geq 1$$

Local HHO dofs

- Recall for second-order PDEs that local HHO dofs comprise
  - cell dofs to approximate the solution in mesh cells
  - face dofs to approximate the solution trace on mesh faces

$$\hat{U}_T := \mathbb{P}^{k+1}(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \quad \text{or} \quad \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \quad k \geq 0$$

- For biharmonic problem, we need additional face dofs
  - either approximating the full gradient trace on mesh faces (vector-valued)
  - or just the normal derivative on mesh faces (scalar-valued)

- The choice studied in [Bonaldi, Di Pietro, Geymonat, Krasucki, 18] is

$$\hat{U}_T := \mathbb{P}^k(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) \times [\mathbb{P}^k(\mathcal{F}_{\partial T})]^d \quad k \geq 1$$

- We consider instead the following two alternatives, both with $k \geq 0$

$$\hat{U}_T := \begin{cases} \mathbb{P}^{k+2}(T) \times \mathbb{P}^{k+1}(\mathcal{F}_{\partial T}) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) & d = 2 \;\rightarrow \text{HHO(A)} \\ \mathbb{P}^{k+2}(T) \times \mathbb{P}^{k+2}(\mathcal{F}_{\partial T}) \times \mathbb{P}^k(\mathcal{F}_{\partial T}) & d \geq 2 \;\rightarrow \text{HHO(B)} \end{cases}$$

Local reconstruction

- Let $T \in \mathcal{T}$
- We want to mimic the integration by parts formula (smooth v, w):

$$(\nabla^2 v, \nabla^2 w)_T = (v, \Delta^2 w)_T - (v, \partial_n \Delta w)_{\partial T} + (\partial_n v, \partial_{nn} w)_{\partial T} + (\partial_t v, \partial_{nt} w)_{\partial T}$$

Local reconstruction

- Let $T \in \mathcal{T}$
- We want to mimic the integration by parts formula (smooth v, w):

$$(\nabla^2 v, \nabla^2 w)_T = (v, \Delta^2 w)_T - (v, \partial_n \Delta w)_{\partial T} + (\partial_n v, \partial_{nn} w)_{\partial T} + (\partial_t v, \partial_{nt} w)_{\partial T}$$

- Let $\hat{v}_T := (v_T, v_{\partial T}, \gamma_{\partial T}) \in \hat{U}_T$
- Potential reconstruction $R_T : \hat{U}_T \longrightarrow \mathbb{P}^{k+2}(T)$ s.t. $\forall w \in \mathbb{P}^{k+2}(T)/\mathbb{P}^1$,

$$(\nabla^2 R_T(\hat{v}_T), \nabla^2 w)_T = (v_T, \Delta^2 w)_T - (v_{\partial T}, \partial_n \Delta w)_{\partial T} + (\gamma_{\partial T}, \partial_{nn} w)_{\partial T} + (\partial_t v_{\partial T}, \partial_{nt} w)_{\partial T}$$

together with $(R_T(\hat{v}_T), \xi)_T = (v_T, \xi)_T$ for all $\xi \in \mathbb{P}^1(T)$

- Hessian reconstruction $\mathcal{H}_T(\hat{v}_T) := \nabla^2 R_T(\hat{v}_T) \in [\mathbb{P}^k(T)]^{d \times d}$

Local stabilization

- The goal of stabilization is to weakly enforce

$$v_T|_{\partial T} \approx v_{\partial T}, \quad \partial_n v_T|_{\partial T} \approx \gamma_{\partial T}, \quad \forall \hat{v}_T := (v_T, v_{\partial T}, \gamma_{\partial T}) \in \hat{U}_T$$

Local stabilization

- The goal of stabilization is to weakly enforce

$$v_T|_{\partial T} \approx v_{\partial T}, \quad \partial_n v_T|_{\partial T} \approx \gamma_{\partial T}, \quad \forall \hat{v}_T := (v_T, v_{\partial T}, \gamma_{\partial T}) \in \hat{U}_T$$

- For HHO(B) with $\hat{U}_T := \mathbb{P}^{k+2}(T) \times \mathbb{P}^{k+2}(\mathcal{F}_{\partial T}) \times \mathbb{P}^{k}(\mathcal{F}_{\partial T})$,

$$S_{\partial T}(\hat{v}_T, \hat{v}_T) := h_T^{-3} \| v_T|_{\partial T} - v_{\partial T} \|_{\partial T}^2 + h_T^{-1} \| \Pi_{\partial T}^k (\partial_n v_T|_{\partial T}) - \gamma_{\partial T} \|_{\partial T}^2$$

$\rightarrow$ natural extension of LS stabilization to biharmonic problem

Local stabilization

- The goal of stabilization is to weakly enforce

$$v_T|_{\partial T} \approx v_{\partial T}, \quad \partial_n v_T|_{\partial T} \approx \gamma_{\partial T}, \quad \forall \hat{v}_T := (v_T, v_{\partial T}, \gamma_{\partial T}) \in \hat{U}_T$$

- For HHO(B) with $\hat{U}_T := \mathbb{P}^{k+2}(T) \times \mathbb{P}^{k+2}(\mathcal{F}_{\partial T}) \times \mathbb{P}^{k}(\mathcal{F}_{\partial T})$,

$$S_{\partial T}(\hat{v}_T, \hat{v}_T) := h_T^{-3} \| v_T|_{\partial T} - v_{\partial T} \|_{\partial T}^2 + h_T^{-1} \| \Pi_{\partial T}^k (\partial_n v_T|_{\partial T}) - \gamma_{\partial T} \|_{\partial T}^2$$

$\rightarrow$ natural extension of LS stabilization to biharmonic problem

- For HHO(A) with $\hat{U}_T := \mathbb{P}^{k+2}(T) \times \mathbb{P}^{k+1}(\mathcal{F}_{\partial T}) \times \mathbb{P}^{k}(\mathcal{F}_{\partial T})$ and $d = 2$

$$S_{\partial T}(\hat{v}_T, \hat{v}_T) := h_T^{-3} \| \Upsilon_{\partial T}^{k+1} (v_T|_{\partial T} - v_{\partial T}) \|_{\partial T}^2 + h_T^{-1} \| \Pi_{\partial T}^k (\partial_n v_T|_{\partial T}) - \gamma_{\partial T} \|_{\partial T}^2$$

where on each face $F \in \mathcal{F}_{\partial T}$, $\Upsilon_{\partial T}^{k+1}$ matches endpoint values and moments on F up to degree $(k-1)$

  - commuting property with tangential derivative (cf. 1D de Rham complex)
  - similar operator available for any $d \geq 2$ but maps onto $\mathbb{P}^{k+d-1}(\mathcal{F}_{\partial T})$

Discrete problem (1/2)

- The local bilinear form writes

$$a_T(\hat{v}_T, \hat{w}_T) := (\nabla^2 R_T(\hat{v}_T), \nabla^2 R_T(\hat{w}_T))_T + S_{\partial T}(\hat{v}_T, \hat{w}_T)$$

Discrete problem (1/2)

- The local bilinear form writes

$$a_T(\hat{v}_T, \hat{w}_T) := (\nabla^2 R_T(\hat{v}_T), \nabla^2 R_T(\hat{w}_T))_T + S_{\partial T}(\hat{v}_T, \hat{w}_T)$$

- Global dofs $\hat{v}_h := (v_{\mathcal{T}}, v_{\mathcal{F}}, \gamma_{\mathcal{F}}) \in \hat{U}_h$ with

$$\hat{U}_h := \mathbb{P}^{k+2}(\mathcal{T}) \times \mathbb{P}^{k+\delta}(\mathcal{F}) \times \mathbb{P}^{k}(\mathcal{F}), \quad \delta \in \{1, 2\}$$

  - all faces oriented by fixed unit normal $\mathbf{n}_F$, γ_F approximates $\mathbf{n}_F \cdot \nabla v$
  - local dofs of $\hat{v}_h$ in a mesh cell $T \in \mathcal{T}$: $(v_T, (v_F)_{F \in \mathcal{F}_{\partial T}}, ((\mathbf{n}_T \cdot \mathbf{n}_F)\gamma_F)_{F \in \mathcal{F}_{\partial T}}$

Discrete problem (1/2)

- The local bilinear form writes

$$a_T(\hat{v}_T, \hat{w}_T) := (\nabla^2 R_T(\hat{v}_T), \nabla^2 R_T(\hat{w}_T))_T + S_{\partial T}(\hat{v}_T, \hat{w}_T)$$

- Global dofs $\hat{v}_h := (v_{\mathcal{T}}, v_{\mathcal{F}}, \gamma_{\mathcal{F}}) \in \hat{U}_h$ with

$$\hat{U}_h := \mathbb{P}^{k+2}(\mathcal{T}) \times \mathbb{P}^{k+\delta}(\mathcal{F}) \times \mathbb{P}^{k}(\mathcal{F}), \quad \delta \in \{1,2\}$$

  - all faces oriented by fixed unit normal $\mathbf{n}_F$, γ_F approximates $\mathbf{n}_F \cdot \nabla v$
  - local dofs of $\hat{v}_h$ in a mesh cell $T \in \mathcal{T}$: $(v_T, (v_F)_{F \in \mathcal{F}_{\partial T}}, ((\mathbf{n}_T \cdot \mathbf{n}_F)\gamma_F)_{F \in \mathcal{F}_{\partial T}}$

- Type I BC's enforced on face boundary dofs by setting $v_F = \gamma_F = 0$ for all $F \subset \partial\Omega \longrightarrow$ subspace $\hat{U}_{h0}$

Discrete problem (2/2)

- Discrete problem: Find $\hat{u}_h \in \hat{U}_{h0}$ s.t.

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T\in\mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) = (f, w_{\mathcal{T}})_\Omega, \qquad \forall \hat{w}_h \in \hat{U}_{h0}$$

Discrete problem (2/2)

- Discrete problem: Find $\hat{u}_h \in \hat{U}_{h0}$ s.t.

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T \in \mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) = (f, w_{\mathcal{T}})_\Omega, \qquad \forall \hat{w}_h \in \hat{U}_{h0}$$

- Cell dofs eliminated locally by static condensation
  - only face dofs are globally coupled
  - cell dofs recovered by local post-processing

Discrete problem (2/2)

- Discrete problem: Find $\hat{u}_h \in \hat{U}_{h0}$ s.t.

$$a_h(\hat{u}_h, \hat{w}_h) := \sum_{T\in\mathcal{T}} a_T(\hat{u}_T, \hat{w}_T) = (f, w_{\mathcal{T}})_\Omega, \qquad \forall \hat{w}_h \in \hat{U}_{h0}$$

- Cell dofs eliminated locally by static condensation
  - only face dofs are globally coupled
  - cell dofs recovered by local post-processing

- Comparison of globally coupled unknowns per mesh interface
  - $d = 2$: $(3k + 3)$ in [Bonaldi et al., 18] vs. $(2k + 3)$ in HHO(A)
  - $d = 3$: $(4k + 4)$ in [Bonaldi et al., 18] vs. $(2k + 4)$ in HHO(B)
  - static condensation is slightly more expensive in HHO(A-B), but cost is compensated by simpler stabilization

- Stability and boundedness: There are $0 < \alpha \le \omega$ s.t. for all $T \in \mathcal{T}$,

$$\alpha \|\hat{v}_T\|^2_{\hat{U}_T} \le a_T(\hat{v}_T, \hat{v}_T) \le \omega \|\hat{v}_T\|^2_{\hat{U}_T}, \quad \forall \hat{v}_T \in \hat{U}_T$$

with $\|\hat{v}_T\|^2_{\hat{U}_T} := \|\nabla^2 v_T\|^2_T + h_T^{-3}\|v_T - v_{\partial T}\|^2_{\partial T} + h_T^{-1}\|\partial_n v_T|_{\partial T} - \gamma_{\partial T}\|^2_{\partial T}$

Stability

- Stability and boundedness: There are $0 < \alpha \leq \omega$ s.t. for all $T \in \mathcal{T}$,

$$\alpha \|\hat{v}_T\|^2_{\hat{U}_T} \leq a_T(\hat{v}_T, \hat{v}_T) \leq \omega \|\hat{v}_T\|^2_{\hat{U}_T}, \quad \forall \hat{v}_T \in \hat{U}_T$$

with $\|\hat{v}_T\|^2_{\hat{U}_T} := \|\nabla^2 v_T\|^2_T + h_T^{-3} \|v_T - v_{\partial T}\|^2_{\partial T} + h_T^{-1} \|\partial_n v_T|_{\partial T} - \gamma_{\partial T}\|^2_{\partial T}$

- $\|\hat{v}_h\|^2_{\hat{U}_h} := \sum_{T \in \mathcal{T}} \|\hat{v}_T\|^2_{\hat{U}_T}$ defines a norm on $\hat{U}_{h0}$
- Discrete problem is well-posed (Lax–Milgram lemma)

Approximation

- Local approximation operator $J_T^{\text{HHO}} : H^2(T) \to \mathbb{P}^{k+2}(T)$

$$J_T^{\text{HHO}} : H^2(T) \xrightarrow{\hat{I}_T} \hat{U}_T \xrightarrow{R_T} \mathbb{P}^{k+2}(T)$$

$$\hat{I}_T(v) := \begin{cases} (\Pi_T^{k+2}(v), \Upsilon_{\partial T}^{k+1}(v|_{\partial T}), \Pi_{\partial T}^{k}(\mathbf{n}_T \cdot \nabla v|_{\partial T})) & \text{for HHO(A)} \\ (\Pi_T^{k+2}(v), \Pi_{\partial T}^{k+2}(v|_{\partial T}), \Pi_{\partial T}^{k}(\mathbf{n}_T \cdot \nabla v|_{\partial T})) & \text{for HHO(B)} \end{cases}$$

Approximation

- Local approximation operator $J_T^{\mathrm{HHO}} : H^2(T) \to \mathbb{P}^{k+2}(T)$

$$J_T^{\mathrm{HHO}} : H^2(T) \xrightarrow{\hat{I}_T} \hat{U}_T \xrightarrow{R_T} \mathbb{P}^{k+2}(T)$$

$$\hat{I}_T(v) := \begin{cases} (\Pi_T^{k+2}(v), \Upsilon_{\partial T}^{k+1}(v|_{\partial T}), \Pi_{\partial T}^{k}(\mathbf{n}_T \cdot \nabla v|_{\partial T})) & \text{for HHO(A)} \\ (\Pi_T^{k+2}(v), \Pi_{\partial T}^{k+2}(v|_{\partial T}), \Pi_{\partial T}^{k}(\mathbf{n}_T \cdot \nabla v|_{\partial T})) & \text{for HHO(B)} \end{cases}$$

- For all $v \in H^{2+s}(T)$, $s > \frac{3}{2}$, set

$$\|v\|_{\sharp,T}^2 := \|\nabla^2 v\|_T + h_T^3 \|\partial_n \Delta v\|_{\partial T}^2 + h_T \|\partial_n \nabla v\|_{\partial T}^2$$

Approximation

- Local approximation operator $J_T^{\mathrm{HHO}} : H^2(T) \to \mathbb{P}^{k+2}(T)$

$$J_T^{\mathrm{HHO}} : H^2(T) \xrightarrow{\hat{I}_T} \hat{U}_T \xrightarrow{R_T} \mathbb{P}^{k+2}(T)$$

$$\hat{I}_T(v) := \begin{cases} (\Pi_T^{k+2}(v), \Upsilon_{\partial T}^{k+1}(v|_{\partial T}), \Pi_{\partial T}^{k}(\mathbf{n}_T\cdot\nabla v|_{\partial T})) & \text{for HHO(A)} \\ (\Pi_T^{k+2}(v), \Pi_{\partial T}^{k+2}(v|_{\partial T}), \Pi_{\partial T}^{k}(\mathbf{n}_T\cdot\nabla v|_{\partial T})) & \text{for HHO(B)} \end{cases}$$

- For all $v \in H^{2+s}(T)$, $s > \frac{3}{2}$, set

$$\|v\|_{\sharp,T}^2 := \|\nabla^2 v\|_T + h_T^3\|\partial_n \Delta v\|_{\partial T}^2 + h_T\|\partial_n \nabla v\|_{\partial T}^2$$

- The following optimal approximation properties hold:

$$\|v - J_T^{\mathrm{HHO}}(v)\|_{\sharp,T} \lesssim \|v - \Pi_T^{k+2}(v)\|_{\sharp,T}$$
$$S_{\partial T}(\hat{I}_T(v), \hat{I}_T(v))^{\frac{1}{2}} \lesssim \|\nabla^2(v - \Pi_T^{k+2}(v))\|_T$$

Moreover, for HHO(A), J_T^{HHO} coincides with the H^2-elliptic projector

Consistency

- Assume exact solution u is in $H^{2+s}(\Omega)$, $s > \frac{3}{2}$

Consistency

- Assume exact solution u is in $H^{2+s}(\Omega)$, $s > \frac{3}{2}$
- Key step when bounding the consistency error: For all $\hat{w}_h \in \hat{U}_{h0}$,

$$
\begin{aligned}
(f, w_{\mathcal{T}})_\Omega &= \sum_{T\in\mathcal{T}} (\Delta^2 u, w_{\mathcal{T}})_\Omega \\
&= \sum_{T\in\mathcal{T}} (\nabla^2 u, \nabla^2 w_{\mathcal{T}})_T + (\partial_n \Delta u, w_{\mathcal{T}})_{\partial T} - (\partial_{nn} u, \partial_n w_{\mathcal{T}})_{\partial T} - (\partial_{nt} u, \partial_t w_{\mathcal{T}})_{\partial T} \\
&= \sum_{T\in\mathcal{T}} (\nabla^2 u, \nabla^2 w_{\mathcal{T}})_T + (\partial_n \Delta u, w_{\mathcal{T}} - w_{\partial T})_{\partial T} - (\partial_{nn} u, \partial_n w_{\mathcal{T}} - \gamma_{\partial T})_{\partial T} - (\partial_{nt} u, \partial_t (w_{\mathcal{T}} - w_{\partial T}))_{\partial T}
\end{aligned}
$$

Consistency

- Assume exact solution u is in $H^{2+s}(\Omega)$, $s > \frac{3}{2}$
- Key step when bounding the consistency error: For all $\hat{w}_h \in \hat{U}_{h0}$,

$$(f, w_{\mathcal{T}})_\Omega = \sum_{T\in\mathcal{T}} (\Delta^2 u, w_{\mathcal{T}})_\Omega$$

$$= \sum_{T\in\mathcal{T}} (\nabla^2 u, \nabla^2 w_{\mathcal{T}})_T + (\partial_n \Delta u, w_{\mathcal{T}})_{\partial T} - (\partial_{nn} u, \partial_n w_{\mathcal{T}})_{\partial T} - (\partial_{nt} u, \partial_t w_{\mathcal{T}})_{\partial T}$$

$$= \sum_{T\in\mathcal{T}} (\nabla^2 u, \nabla^2 w_{\mathcal{T}})_T + (\partial_n \Delta u, w_{\mathcal{T}} - w_{\partial T})_{\partial T} - (\partial_{nn} u, \partial_n w_{\mathcal{T}} - \gamma_{\partial T})_{\partial T} - (\partial_{nt} u, \partial_t (w_{\mathcal{T}} - w_{\partial T}))_{\partial T}$$

- Then, letting $\chi(\hat{w}_h) := (f, w_{\mathcal{T}})_\Omega - a_h(\hat{I}_{\mathcal{T}}(u), \hat{w}_h)$, we obtain

$$|\chi(\hat{w}_h)| \lesssim \|\eta\|_{\sharp,\mathcal{T}} \|\hat{w}_h\|_{\hat{U}_h}, \qquad \eta|_T := u|_T - J_T^{\mathrm{HHO}}(u)$$

and $\|\eta\|_{\sharp,\mathcal{T}}$ is bounded by $\|u - \Pi_{\mathcal{T}}^{k+2}(u)\|_{\sharp,\mathcal{T}}$ (with $\|\cdot\|_{\sharp,\mathcal{T}}^2 := \sum_{T\in\mathcal{T}} \|\cdot\|_{\sharp,T}^2$)

Error estimate

- Recall assumption $u \in H^{2+s}(\Omega)$, $s > \frac{3}{2}$
- The following error estimate holds:

$$\|\nabla^2_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_{\Omega} \lesssim \|u - \Pi^{k+2}_{\mathcal{T}}(u)\|_{\sharp,\mathcal{T}}$$

Error estimate

- Recall assumption $u \in H^{2+s}(\Omega)$, $s > \frac{3}{2}$
- The following error estimate holds:

$$\|\nabla^2_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_\Omega \lesssim \|u - \Pi^{k+2}_{\mathcal{T}}(u)\|_{\sharp,\mathcal{T}}$$

- If $k \geq 1$ and $u \in H^{k+3}(\Omega)$, $\|\nabla^2_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_\Omega \lesssim h^{k+1}|u|_{H^{k+3}}$
- If $k = 0$, $\|\nabla^2_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_\Omega \lesssim h(|u|_{H^3} + h^\sigma |u|_{H^{3+\sigma}})$, $\sigma := \min(s-1, 1)$

Error estimate

- Recall assumption $u \in H^{2+s}(\Omega)$, $s > \frac{3}{2}$
- The following error estimate holds:

$$\|\nabla^2_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_{\Omega} \lesssim \|u - \Pi^{k+2}_{\mathcal{T}}(u)\|_{\sharp,\mathcal{T}}$$

- If $k \geq 1$ and $u \in H^{k+3}(\Omega)$, $\|\nabla^2_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_{\Omega} \lesssim h^{k+1}|u|_{H^{k+3}}$
- If $k = 0$, $\|\nabla^2_{\mathcal{T}}(u - R_{\mathcal{T}}(\hat{u}_h))\|_{\Omega} \lesssim h(|u|_{H^3} + h^{\sigma}|u|_{H^{3+\sigma}})$, $\sigma := \min(s-1, 1)$
- Circumventing regularity assumption
  - [Veeser, Zanotti, 18-19] for Morley element and C^0-IPDG ($f \in H^{-2}(\Omega)$ in 2D); extension to 3D with arbitrary degree not obvious
  - [Carstensen, Nataraj, 21] for further results on lowest-order methods
  - it is also possible to extend the techniques of [AE, Guermond, 21 (FoCM)] $\Longrightarrow$ allow for any $s > 1$ (and even $s > 0$ for type II BC's)

Literature overview

- Comparison with WG
  - WG are designed using suboptimal plain least-squares stabilization
  - in the table, all the methods deliver $O(h^{k+1})$ H^2-error estimate

method	cell	face	grad	k	ref.
WG	$k+2$	$k+2$	$[k+1]^d$	$k \geq 0$	[Mu, Wang, Ye, 14]
	$k+2$	$k+2$	$k+1$	$k \geq 0$	[Mu, Wang, Ye, 14]
	$k+2$	$k+1$	$k+1$	$k \geq 0$	[Zhang, Zhai, 15]
	1	1	$[1]^d$	$k = 0$	[Ye, Zhang, Zhang, 20]
HHO	k	k	$[k]^d$	$k \geq 1$	[Bonaldi et al., 18]
HHO(A)	$k+2$	$k+1$	k	$k \geq 0$	present ($d = 2$)
HHO(B)	$k+2$	$k+2$	k	$k \geq 0$	present ($d \geq 2$)

Literature overview

- Comparison with WG
  - WG are designed using suboptimal plain least-squares stabilization
  - in the table, all the methods deliver $O(h^{k+1})$ H^2-error estimate

method	cell	face	grad	k	ref.
WG	$k+2$	$k+2$	$[k+1]^d$	$k \geq 0$	[Mu, Wang, Ye, 14]
	$k+2$	$k+2$	$k+1$	$k \geq 0$	[Mu, Wang, Ye, 14]
	$k+2$	$k+1$	$k+1$	$k \geq 0$	[Zhang, Zhai, 15]
	1	1	$[1]^d$	$k = 0$	[Ye, Zhang, Zhang, 20]
HHO	k	k	$[k]^d$	$k \geq 1$	[Bonaldi et al., 18]
HHO(A)	$k+2$	$k+1$	k	$k \geq 0$	present ($d = 2$)
HHO(B)	$k+2$	$k+2$	k	$k \geq 0$	present ($d \geq 2$)

- Broader literature review
  - C^1-VEM [Brezzi, Marini, 13; Chinosi, Marini, 16; Antonietti, Manzini, Verani, 20], C^0-VEM [Zhao, Chen, Zhang, 16]
  - DG [Mozolevski, Süli, 03; Georgoulis, Houston, 09], C^0-IPDG [Engel et al., 02; Brenner, Sung, 05]

Further topics

- Nitsche's method and curved boundaries
  - extends ideas from [Burman, AE, 18; Burman, Cicuttin, Delay, AE, 21] on second-order (interface) problems
  - key idea: discard integrals on $\partial\Omega$ when building reconstruction operator
  - boundary-penalty term needs $O(1)$ coefficient

Further topics

- Nitsche's method and curved boundaries
  - extends ideas from [Burman, AE, 18; Burman, Cicuttin, Delay, AE, 21] on second-order (interface) problems
  - key idea: discard integrals on $\partial\Omega$ when building reconstruction operator
  - boundary-penalty term needs $O(1)$ coefficient

- Singular perturbation

$$-\Delta u + \varepsilon \Delta^2 u = f, \qquad \varepsilon \geq 0$$

  - use local cutoff function $\sigma_T = \max(1, \varepsilon h_T^{-2})$ to weigh stabilization terms
  - method and analysis fully robust up to $\varepsilon = 0$

Further topics

- Nitsche's method and curved boundaries
  - extends ideas from [Burman, AE, 18; Burman, Cicuttin, Delay, AE, 21] on second-order (interface) problems
  - key idea: discard integrals on $\partial\Omega$ when building reconstruction operator
  - boundary-penalty term needs $O(1)$ coefficient

- Singular perturbation

$$-\Delta u + \varepsilon \Delta^2 u = f, \qquad \varepsilon \geq 0$$

  - use local cutoff function $\sigma_T = \max(1, \varepsilon h_T^{-2})$ to weigh stabilization terms
  - method and analysis fully robust up to $\varepsilon = 0$

- C^0-HHO: an extension of C^0-FEM!
  - restrict to simplicial/quad/hex meshes
  - local dofs related to the solution trace no longer needed

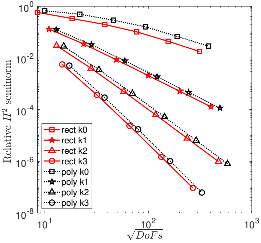
$$\hat{U}_T := \mathbb{P}^{k+2}(T) \times \mathbb{P}^k(\mathcal{F}_{\partial T})$$

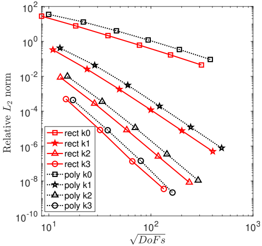
  - error analysis proceeds as above

Numerical results

Convergence rates

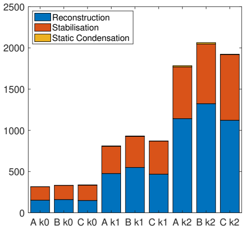
- Smooth solution $u(x, y) = \sin(\pi x)^2 \sin(\pi y)^2$
- HHO(A), $k \in \{0, 1, 2, 3\}$, rectangular and polygonal (Voronoi) meshes
- Left: H^2-seminorm, $O(h^{k+1})$
- Right: L^2-norm, $O(h^{k+3})$ for $k \geq 1$ and $O(h^2)$ for $k = 0$

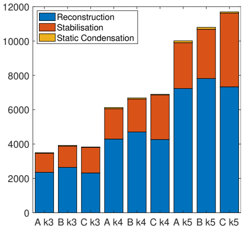




Computational times

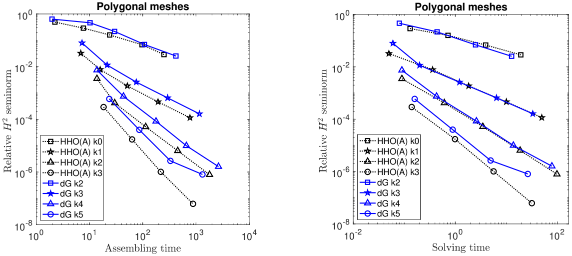
- Time spent on reconstruction, stabilization and static condensation
- Comparison of HHO(A), HHO(B), and HHO(C) which uses reconstruction in stabilization
- $k \in \{0, \ldots, 5\}$, polygonal mesh with 16k cells
- HHO(A) is the most efficient method





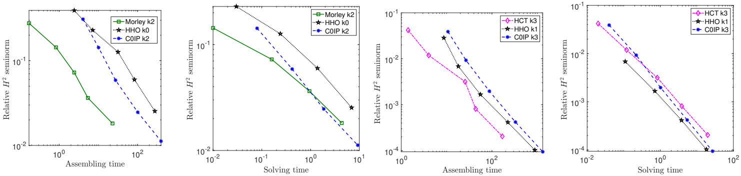
Comparison with DG

- HHO(A) and DG on polygonal mesh (16k cells)
- $k \in \{0, 1, 2, 3\}$ for HHO(A) and $\ell = k + 2$ for DG
- Disclaimer: simple `Matlab` implementation, no optimization
- Some (preliminary) comments
    - HHO leads to less dofs and lower assembling time than DG (cell dofs richer than face ones; numerical DG fluxes longer to evaluate)
    - solving time smaller for DG for $k \leq 2$ and smaller for HHO if $k \geq 3$ (HHO stencil less compact than DG stencil)

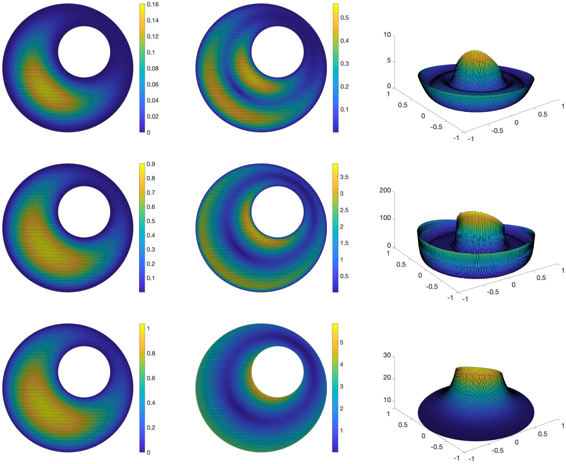


Comparison with Morley, HCT and C^0-IPDG

- Triangular meshes, finest one has 32k cells & 49k edges
- All compared methods deliver same decay rate on H^2-error
- Morley FEM more efficient than HHO($k = 0$)
- HCT FEM more efficient than HHO($k = 1$) if assembling time is considered, but not if solving time is considered
- HHO(k) more efficient than C^0-IPDG($k + 2$), $k \in \{0, 1\}$



- Triangular mesh composed of 9.4k cells, $k = 1$
- From top to bottom: $\varepsilon = 1$, $\varepsilon = 10^{-3}$, $\varepsilon = 0(!)$
- From left to right: solution, gradient, Hessian (reconstructed)



Error analysis with low regularity

Localizing normal traces

- Brief summary of [AE, Guermond 21 (FoCM & *Finite Elements*, Chaps. 40-41)]
- Let $p > 2$ and $q \in (\frac{2d}{d+2}, 2]$
- There is $\rho \in (2, p]$ s.t. $q \geq \frac{\rho d}{\rho + d}$; let $\rho' \in [p', 2)$ s.t. $\frac{1}{\rho} + \frac{1}{\rho'} = 1$

Localizing normal traces

- Brief summary of [AE, Guermond 21 (FoCM & *Finite Elements*, Chaps. 40-41)]
- Let $p > 2$ and $q \in (\frac{2d}{d+2}, 2]$
- There is $\rho \in (2, p]$ s.t. $q \geq \frac{\rho d}{\rho+d}$; let $\rho' \in [p', 2)$ s.t. $\frac{1}{\rho} + \frac{1}{\rho'} = 1$
- For all $T \in \mathcal{T}$ and all $F \in \mathcal{F}_{\partial T}$, consider

$$L_F^T : W^{\frac{1}{\rho},\rho'}(F) \xrightarrow{\text{zero extension}} W^{\frac{1}{\rho},\rho'}(\partial T) \xrightarrow{\text{trace lifting}} W^{1,\rho'}(T)$$

Localizing normal traces

- Brief summary of [AE, Guermond 21 (FoCM & *Finite Elements*, Chaps. 40-41)]
- Let $p > 2$ and $q \in (\frac{2d}{d+2}, 2]$
- There is $\rho \in (2, p]$ s.t. $q \geq \frac{\rho d}{\rho + d}$; let $\rho' \in [p', 2)$ s.t. $\frac{1}{\rho} + \frac{1}{\rho'} = 1$
- For all $T \in \mathcal{T}$ and all $F \in \mathcal{F}_{\partial T}$, consider

$$L_F^T : W^{\frac{1}{\rho},\rho'}(F) \xrightarrow{\text{zero extension}} W^{\frac{1}{\rho},\rho'}(\partial T) \xrightarrow{\text{trace lifting}} W^{1,\rho'}(T)$$

- Let $\sigma \in \mathbf{S}^{\mathrm{d}}(T) := \{\tau \in \mathbf{L}^p(T); \nabla\cdot\tau \in L^q(T)\}$ ($^{\mathrm{d}}$ stands for divergence)
- Define $\gamma_{T,F}^{\mathrm{d}}(\sigma) \in (W^{\frac{1}{\rho},\rho'}(F))'$ s.t. for all $\phi \in W^{\frac{1}{\rho},\rho'}(F)$,

$$\langle \gamma_{T,F}^{\mathrm{d}}(\sigma), \phi \rangle_F := \int_T \{\sigma\cdot\nabla L_F^T(\phi) + (\nabla\cdot\sigma) L_F^T(\phi)\}$$

If σ is smooth, $\gamma_{T,F}^{\mathrm{d}}(\sigma) = (\sigma\cdot\mathbf{n}_T)|_F$

Poisson problem with DG (1/2)

- Assume $u \in V_\sharp := \{v \in H^{1+s}(\Omega); \Delta v \in L^q(\Omega)\}$, $s > 0$
- For all $v \in V_\sharp$, $\nabla v \in \mathbf{H}^s(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$, $p > 2$; hence,

$$\nabla v \in \mathbf{S}^{\mathrm{d}}(\Omega) := \{\sigma \in \mathbf{L}^p(\Omega); \nabla\cdot\sigma \in L^q(\Omega)\}$$

Poisson problem with DG (1/2)

- Assume $u \in V_\sharp := \{v \in H^{1+s}(\Omega); \Delta v \in L^q(\Omega)\}$, $s > 0$
- For all $v \in V_\sharp$, $\nabla v \in \mathbf{H}^s(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$, $p > 2$; hence,

$$\nabla v \in \mathbf{S}^{\mathrm{d}}(\Omega) := \{\sigma \in \mathbf{L}^p(\Omega); \nabla\cdot\sigma \in L^q(\Omega)\}$$

- Bilinear form on $(V_\sharp + \mathbb{P}^k(\mathcal{T})) \times \mathbb{P}^k(\mathcal{T})$

$$n_\sharp^{(2)}(v, w_\mathcal{T}) := \sum_{T\in\mathcal{T}} \sum_{F\in\mathcal{F}_{\partial T}} \langle \gamma_{T,F}^{\mathrm{d}}(\nabla v), w_T|_F - \{w_\mathcal{T}\}_F \rangle_F$$

Notice that $n_\sharp^{(2)}(v_\mathcal{T}, w_\mathcal{T}) = \sum_{F\in\mathcal{F}} \int_F \{\nabla v_\mathcal{T}\}_F \cdot \mathbf{n}_F [\![w_\mathcal{T}]\!]_F$ if $v_\mathcal{T} \in \mathbb{P}^k(\mathcal{T})$

Poisson problem with DG (1/2)

- Assume $u \in V_\sharp := \{v \in H^{1+s}(\Omega); \Delta v \in L^q(\Omega)\}$, $s > 0$
- For all $v \in V_\sharp$, $\nabla v \in \mathbf{H}^s(\Omega) \hookrightarrow \mathbf{L}^p(\Omega)$, $p > 2$; hence,

$$\nabla v \in \mathbf{S}^{\mathrm{d}}(\Omega) := \{\sigma \in \mathbf{L}^p(\Omega); \nabla{\cdot}\sigma \in L^q(\Omega)\}$$

- Bilinear form on $(V_\sharp + \mathbb{P}^k(\mathcal{T})) \times \mathbb{P}^k(\mathcal{T})$

$$n_\sharp^{(2)}(v, w_\mathcal{T}) := \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}_{\partial T}} \langle \gamma^{\mathrm{d}}_{T,F}(\nabla v), w_T|_F - \{w_\mathcal{T}\}_F \rangle_F$$

Notice that $n_\sharp^{(2)}(v_\mathcal{T}, w_\mathcal{T}) = \sum_{F \in \mathcal{F}} \int_F \{\nabla v_\mathcal{T}\}_F {\cdot} \mathbf{n}_F [\![w_\mathcal{T}]\!]_F$ if $v_\mathcal{T} \in \mathbb{P}^k(\mathcal{T})$

- Using commuting mollification operators, one proves that for all $v \in V_\sharp$,

$$n_\sharp^{(2)}(v, w_\mathcal{T}) = \sum_{T \in \mathcal{T}} (\nabla v, \nabla w_T)_T + (\Delta v, w_T)_T$$

This property essentially appears as an assumption in the medius analysis [Gudi, 10]

Poisson problem with DG (2/2)

- Consider interior penalty DG (IPDG) [Arnold, 82]
- The key relation for consistency is

$$(f, w_{\mathcal{T}})_\Omega = \sum_{T\in\mathcal{T}} (-\Delta u, w_T)_T = \sum_{T\in\mathcal{T}} (\nabla u, \nabla w_T)_T - n_\sharp^{(2)}(u, w_{\mathcal{T}})$$

Poisson problem with DG (2/2)

- Consider interior penalty DG (IPDG) [Arnold, 82]
- The key relation for consistency is

$$(f, w_{\mathcal{T}})_\Omega = \sum_{T\in\mathcal{T}} (-\Delta u, w_T)_T = \sum_{T\in\mathcal{T}} (\nabla u, \nabla w_T)_T - n_\sharp^{(2)}(u, w_{\mathcal{T}})$$

- For IPDG, $a_{\mathcal{T}}^{\mathrm{DG}}(v_{\mathcal{T}}, w_{\mathcal{T}}) = \sum_{T\in\mathcal{T}} (\nabla v_T, \nabla w_T)_T - n_\sharp^{(2)}(v_{\mathcal{T}}, w_{\mathcal{T}}) + \text{stb}.$

Poisson problem with DG (2/2)

- Consider interior penalty DG (IPDG) [Arnold, 82]
- The key relation for consistency is

$$(f, w_{\mathcal{T}})_\Omega = \sum_{T\in\mathcal{T}} (-\Delta u, w_{\mathcal{T}})_T = \sum_{T\in\mathcal{T}} (\nabla u, \nabla w_T)_T - n_\sharp^{(2)}(u, w_{\mathcal{T}})$$

- For IPDG, $a_{\mathcal{T}}^{\mathrm{DG}}(v_{\mathcal{T}}, w_{\mathcal{T}}) = \sum_{T\in\mathcal{T}} (\nabla v_T, \nabla w_T)_T - n_\sharp^{(2)}(v_{\mathcal{T}}, w_{\mathcal{T}}) + \text{stb.}$
- Letting $\eta := u - \Pi_{\mathcal{T}}^k(u)$, the consistency error is bounded as follows:

$$\begin{aligned}\chi(w_{\mathcal{T}}) &:= (f, w_{\mathcal{T}})_\Omega - a_{\mathcal{T}}^{\mathrm{DG}}(\Pi_{\mathcal{T}}^k(u), w_{\mathcal{T}}) \\ &= \sum_{T\in\mathcal{T}} (\nabla \eta, \nabla w_T)_T - n_\sharp^{(2)}(\eta, w_{\mathcal{T}}) + \text{stb.}\end{aligned}$$

Conclude with boundedness property $|n_\sharp^{(2)}(\eta, w_{\mathcal{T}})| \lesssim \|\eta\|_{\sharp,\mathcal{T}} \|w_{\mathcal{T}}\|_h$

Adaptation to HHO

- Exploiting the face variable representing the trace, we define the following bilinear form on $(V_\sharp + \mathbb{P}^{k+1}(\mathcal{T})) \times \hat{U}_{h0}$:

$$\hat{n}_\sharp^{(2)}(v, \hat{w}_h) := \sum_{T\in\mathcal{T}} \sum_{F\in\mathcal{F}_{\partial T}} \langle \gamma_{T,F}^{\mathrm{d}}(\nabla v), w_T|_F - w_{\partial T}|_F \rangle_F$$

Adaptation to HHO

- Exploiting the face variable representing the trace, we define the following bilinear form on $(V_\sharp + \mathbb{P}^{k+1}(\mathcal{T})) \times \hat{U}_{h0}$:

$$\hat{n}_\sharp^{(2)}(v, \hat{w}_h) := \sum_{T \in \mathcal{T}} \sum_{F \in \mathcal{F}_{\partial T}} \langle \gamma_{T,F}^{\mathrm{d}}(\nabla v), w_T|_F - w_{\partial T}|_F \rangle_F$$

- The first key relation is $\hat{n}_\sharp^{(2)}(v, \hat{w}_h) = n_\sharp^{(2)}(v, w_{\mathcal{T}})$ for all $v \in V_\sharp$

Adaptation to HHO

- Exploiting the face variable representing the trace, we define the following bilinear form on $(V_\sharp + \mathbb{P}^{k+1}(\mathcal{T})) \times \hat{U}_{h0}$:

$$\hat{n}_\sharp^{(2)}(v, \hat{w}_h) := \sum_{T\in\mathcal{T}} \sum_{F\in\mathcal{F}_{\partial T}} \langle \gamma^{\mathrm{d}}_{T,F}(\nabla v), w_T|_F - w_{\partial T}|_F \rangle_F$$

- The first key relation is $\hat{n}_\sharp^{(2)}(v, \hat{w}_h) = n_\sharp^{(2)}(v, w_{\mathcal{T}})$ for all $v \in V_\sharp$
- The link to the reconstruction operator is as follows:

$$\begin{aligned} a_h(\hat{I}_{\mathcal{T}}(u), \hat{w}_h) &= \sum_{T\in\mathcal{T}} (\nabla J_T^{\mathrm{HHO}}(u), \nabla R_T(\hat{w}_T))_T + \text{stb.} \\ &= \sum_{T\in\mathcal{T}} (\nabla J_T^{\mathrm{HHO}}(u), \nabla w_T)_T - \hat{n}_\sharp^{(2)}(J_{\mathcal{T}}^{\mathrm{HHO}}(u), \hat{w}_h) + \text{stb.} \end{aligned}$$

Adaptation to HHO

- Exploiting the face variable representing the trace, we define the following bilinear form on $(V_\sharp + \mathbb{P}^{k+1}(\mathcal{T})) \times \hat{U}_{h0}$:

$$\hat{n}_\sharp^{(2)}(v, \hat{w}_h) := \sum_{T\in\mathcal{T}} \sum_{F\in\mathcal{F}_{\partial T}} \langle \gamma_{T,F}^{\mathrm{d}}(\nabla v), w_T|_F - w_{\partial T}|_F \rangle_F$$

- The first key relation is $\hat{n}_\sharp^{(2)}(v, \hat{w}_h) = n_\sharp^{(2)}(v, w_\mathcal{T})$ for all $v \in V_\sharp$
- The link to the reconstruction operator is as follows:

$$\begin{aligned} a_h(\hat{I}_\mathcal{T}(u), \hat{w}_h) &= \sum_{T\in\mathcal{T}} (\nabla J_T^{\mathrm{HHO}}(u), \nabla R_T(\hat{w}_T))_T + \text{stb.} \\ &= \sum_{T\in\mathcal{T}} (\nabla J_T^{\mathrm{HHO}}(u), \nabla w_T)_T - \hat{n}_\sharp^{(2)}(J_\mathcal{T}^{\mathrm{HHO}}(u), \hat{w}_h) + \text{stb.} \end{aligned}$$

- Letting $\eta := u - J_\mathcal{T}^{\mathrm{HHO}}(u)$, we recover

$$\chi(\hat{w}_h) = \sum_{T\in\mathcal{T}} (\nabla \eta, w_T)_T - \hat{n}_\sharp^{(2)}(\eta, \hat{w}_h) + \text{stb.}$$

Biharmonic problem

- Above technique extends to IPDG/HHO for biharmonic problem
- The critical step is to give a meaning to $\partial_n \Delta v$ on mesh faces

Biharmonic problem

- Above technique extends to IPDG/HHO for biharmonic problem
- The critical step is to give a meaning to $\partial_n \Delta v$ on mesh faces
- If $u \in H^{3+s}(\Omega)$, $s > 0$, and $f \in L^q(\Omega)$, $q \in (\frac{2d}{2+d}, 2]$, then

$$\sigma := \nabla \Delta u \in \mathbf{S}^{\mathrm{d}}(\Omega)$$

$\Longrightarrow \gamma^{\mathrm{d}}_{T,F}(\sigma)$ is well defined on all the mesh faces

C^0-methods with type II BC's (1/2)

- In this setting, we can lower the regularity even further

$$u \in H^{2+s}(\Omega),\ s > 0, \qquad f \in H^{-1}(\Omega)$$

With type II BC's, one has $\Delta u \in H_0^1(\Omega) \Longrightarrow \nabla \Delta u \in \mathbf{L}^2(\Omega)$!

- In this setting, we can lower the regularity even further

$$u \in H^{2+s}(\Omega),\ s > 0, \qquad f \in H^{-1}(\Omega)$$

With type II BC's, one has $\Delta u \in H^1_0(\Omega) \Longrightarrow \nabla\Delta u \in \mathbf{L}^2(\Omega)$!

- Let us set $V_\sharp := \{v \in H^{2+s}(\Omega); \Delta v \in H^1_0(\Omega)\}$
- In C^0-HHO, the cell dofs are in $\mathbb{P}^{g,k}(\mathcal{T}) := \mathbb{P}^k(\mathcal{T}) \cap H^1_0(\Omega)$

C^0-methods with type II BC's (1/2)

- In this setting, we can lower the regularity even further

$$u \in H^{2+s}(\Omega),\ s > 0, \qquad f \in H^{-1}(\Omega)$$

With type II BC's, one has $\Delta u \in H^1_0(\Omega) \Longrightarrow \nabla\Delta u \in \mathbf{L}^2(\Omega)$!

- Let us set $V_\sharp := \{v \in H^{2+s}(\Omega); \Delta v \in H^1_0(\Omega)\}$
- In C^0-HHO, the cell dofs are in $\mathbb{P}^{\mathrm{g},k}(\mathcal{T}) := \mathbb{P}^k(\mathcal{T}) \cap H^1_0(\Omega)$
- We consider on $(V_\sharp \times \mathbb{P}^{\mathrm{g},k}(\mathcal{T})) \times \mathbb{P}^{\mathrm{g},k}(\mathcal{T})$ the bilinear form

$$n_\sharp^{(4)}(v, w_\mathcal{T}) := \sum_{T\in\mathcal{T}} \sum_{F\in\mathcal{F}_{\partial T}} \sum_{i\in\{1:d\}} \langle \gamma^{\mathrm{d}}_{T,F}(\nabla\partial_i v), \mathbf{n}_{T,i}(\partial_n w_T - \mathbf{n}_T\cdot\{\nabla w_\mathcal{T}\}_F)|_F\rangle_F$$

Notice that $\nabla\partial_i v \in \mathbf{S}^{\mathrm{d}}(\Omega)$ for all $i \in \{1:d\}$ (with $q = 2$)

C^0-methods with type II BC's (2/2)

- The key relation for consistency in C^0-IPDG is

$$\langle \Delta^2 u, w_{\mathcal{T}} \rangle_{H^{-1},H_0^1} = \sum_{T\in\mathcal{T}} (\nabla^2 u, \nabla^2 w_T)_T - n_\sharp^{(4)}(u, w_{\mathcal{T}})$$

C^0-methods with type II BC's (2/2)

- The key relation for consistency in C^0-IPDG is

$$\langle \Delta^2 u, w_{\mathcal{T}} \rangle_{H^{-1},H_0^1} = \sum_{T\in\mathcal{T}} (\nabla^2 u, \nabla^2 w_T)_T - n_\sharp^{(4)}(u, w_{\mathcal{T}})$$

- For C^0-HHO, one exploits the presence of the face variable representing the normal derivative by setting

$$\hat{n}_\sharp^{(4)}(v, \hat{w}_h) := \sum_{i\in\{1:d\}} \sum_{T\in\mathcal{T}} \sum_{F\in\mathcal{F}_{\partial T}} \langle \gamma_{T,F}^{\mathrm{d}}(\nabla\partial_i v), \mathbf{n}_{T,i}(\partial_n w_T - \chi_{\partial T})|_F \rangle_F$$

C^0-methods with type II BC's (2/2)

- The key relation for consistency in C^0-IPDG is

$$\langle \Delta^2 u, w_{\mathcal{T}} \rangle_{H^{-1},H_0^1} = \sum_{T\in\mathcal{T}} (\nabla^2 u, \nabla^2 w_T)_T - n_\sharp^{(4)}(u, w_{\mathcal{T}})$$

- For C^0-HHO, one exploits the presence of the face variable representing the normal derivative by setting

$$\hat{n}_\sharp^{(4)}(v, \hat{w}_h) := \sum_{i\in\{1:d\}} \sum_{T\in\mathcal{T}} \sum_{F\in\mathcal{F}_{\partial T}} \langle \gamma_{T,F}^{\mathrm{d}}(\nabla \partial_i v), \mathbf{n}_{T,i}(\partial_n w_T - \chi_{\partial T})|_F \rangle_F$$

- The link to the reconstruction operator is as follows:

$$a_h(\hat{I}_{\mathcal{T}}(u), \hat{w}_h) = \sum_{T\in\mathcal{T}} (\nabla^2 J_T^{\mathrm{HHO}}(u), \nabla^2 w_T)_T - \hat{n}_\sharp^{(4)}(J_{\mathcal{T}}^{\mathrm{HHO}}(u), \hat{w}_h) + \mathrm{stb}.$$

Moreover,

$$\langle \Delta^2 u, w_{\mathcal{T}} \rangle_{H^{-1},H_0^1} = \sum_{T\in\mathcal{T}} (\nabla^2 u, \nabla^2 w_T)_T - \hat{n}_\sharp^{(4)}(u, \hat{w}_h)$$

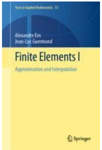
Some references

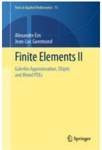
- HHO
  - [Di Pietro, AE, Lemaire 14 (CMAM); Di Pietro, AE 15 (CMAME)]
- HHO for biharmonic problem
  - [Bonaldi et al. 18 (M2AN)]
  - [Dong & AE 21 (hal-03185683); 21 (M2AN)]
- Error analysis with low regularity [AE, Guermond 21 (FoCM)]

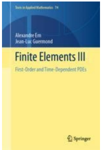
Some references

- HHO
  - [Di Pietro, AE, Lemaire 14 (CMAM); Di Pietro, AE 15 (CMAME)]
- HHO for biharmonic problem
  - [Bonaldi et al. 18 (M2AN)]
  - [Dong & AE 21 (hal-03185683); 21 (M2AN)]
- Error analysis with low regularity [AE, Guermond 21 (FoCM)]
- Recent Finite Element book(s) (Springer, TAM vols. 72-74, 2021)

  with J.-L. Guermond, 83 chapters of 12/14 pages plus about 500 exercises



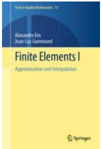


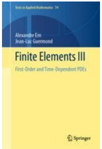


Some references

- HHO
  - [Di Pietro, AE, Lemaire 14 (CMAM); Di Pietro, AE 15 (CMAME)]
- HHO for biharmonic problem
  - [Bonaldi et al. 18 (M2AN)]
  - [Dong & AE 21 (hal-03185683); 21 (M2AN)]
- Error analysis with low regularity [AE, Guermond 21 (FoCM)]
- Recent Finite Element book(s) (Springer, TAM vols. 72-74, 2021)

  with J.-L. Guermond, 83 chapters of 12/14 pages plus about 500 exercises





Thank you for your attention!