Finite element approximation of nonlinear elliptic systems with linear growth

(strain-limiting elastic models)

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Joint work with

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Implicit constitutive theory & strain-limiting models

During the past decade there has been significant progress in developing implicit constitutive models for the description of nonlinear responses of materials. The field was initiated by K.R. Rajagopal (Texas A&M):

K. R. Rajagopal, On implicit constitutive theories.
Appl. Math., 48 (2003), 279–319.

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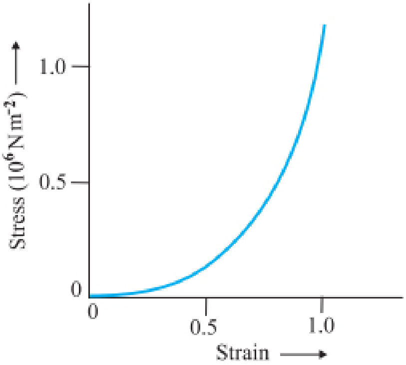
Within "Rajagopal elasticity" it is possible to have models in which the linearized strain is a bounded function, even when the stress is large.

K. R. Rajagopal, Non-linear elastic bodies exhibiting limiting small strain.
Math. Mech. Solids 16 (2011), no. 1, 122–139.

Soft tissue exhibiting finite extensibility



Stress–strain curve for the tissue of the cardiac aorta.

Source:

https://www.toppr.com/guides/physics/mechanical-properties-of-solids/hookes-law-and-stress-strain-curve/

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They may be useful in modelling stress concentration when the displacement gradient is relatively small (e.g. brittle materials near crack tips/notches, or concentrated loads inside a body or on its boundary).

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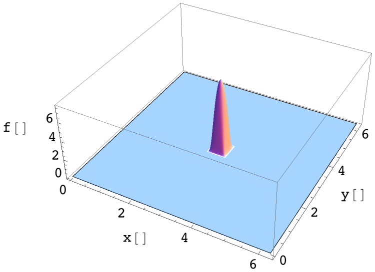
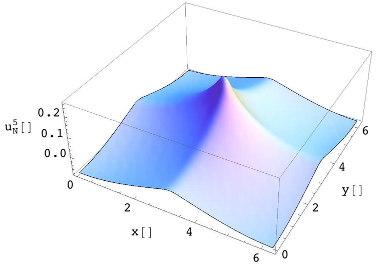


(a) Concentrated load



(b) Small displacement

N. Gelmetti and E. Süli. Spectral approximation of a strain-limiting nonlinear elastic model. Matematički Vesnik. 71, 1-2 (2019), 63–89.

For physical aspects and results concerning the existence of solutions, see:

- M. Bulíček, J. Málek, K. R. Rajagopal, E. Süli, On elastic solids with limiting small strain: modelling and analysis, EMS Surveys in Mathematical Sciences, 1(2) (2014), 283–332. Henceforth: [BMRS2014]
- M. Bulíček, J. Málek, E. Süli, Analysis and approximation of a strain-limiting nonlinear model, Math. Mech. Solids, 20 (2015), 92–118.
- L. Beck, M. Bulíček, J. Málek, E. Süli, On the existence of integrable solutions to nonlinear elliptic systems and variational problems with linear growth, Arch. Ration. Mech. Anal., 225(2) (2017), 717–769. Henceforth: [BBMS2017]

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The literature on the analysis of numerical method for these is very limited:

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- A. Bonito, V. Girault, E. Süli, Finite element approximation of a strain-limiting elastic model, IMA J. Numer. Anal., V. 40, Issue 1, (2020), 29–86.

- A. Bonito, V. Girault, D. Guignard, K. Rajagopal, and E. Süli. Finite element approximation of steady flows of colloidal solutions. ESAIM M2AN, Vol. 55, No. 5, September-October 2021, pp. 1963–2011.

Statement of the model

On a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and for a given external force $\mathbf{f} : \Omega \to \mathbb{R}^d$, we consider the nonlinear elastic model

$$-\operatorname{div}(\mathbf{T}) = \mathbf{f} \qquad \text{in } \Omega, \tag{1}$$

where the stress tensor $\mathbf{T}$ is related to the linearized strain tensor

$$\varepsilon(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}),$$

for a given displacement vector $\mathbf{u}$, via the nonlinear constitutive relation

$$\varepsilon(\mathbf{u}) = \lambda(\operatorname{tr}(\mathbf{T}))\operatorname{tr}(\mathbf{T})\mathbf{I} + \mu(|\mathbf{T}^{\mathbf{d}}|)\,\mathbf{T}^{\mathbf{d}} \qquad \text{in } \Omega. \tag{2}$$

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Here $\lambda \in \mathscr{C}^0(\mathbb{R})$ and $\mu \in \mathscr{C}^0([0,+\infty))$ are given functions and $\mathbf{T}^{\mathbf{d}}$ is the deviatoric part of the tensor $\mathbf{T}$, defined by

$$\mathbf{T}^{\mathbf{d}} := \mathbf{T} - \frac{1}{d}\operatorname{tr}(\mathbf{T})\mathbf{I}.$$

Assume that $s \in \mathbb{R} \mapsto \lambda(s)s \in \mathscr{C}^1(\mathbb{R})$, and that λ and μ satisfy, for some positive constants C_1, C_2, κ and α, the following inequalities:

$$\frac{C_1 s^2}{\kappa + |s|} \leq \lambda(s)s^2 \leq C_2|s| \qquad \forall\, s \in \mathbb{R}; \tag{A1}$$

$$\frac{C_1 s^2}{\kappa + s} \leq \mu(s)s^2 \leq C_2 s \qquad \forall\, s \in \mathbb{R}_{\geq 0}; \tag{A2}$$

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}s}(\lambda(s)s) \qquad \forall\, s \in \mathbb{R}; \tag{A3}$$

$$\frac{C_1}{(\kappa + s)^{\alpha+1}} \leq \frac{\mathrm{d}}{\mathrm{d}s}(\mu(s)s) \qquad \forall\, s \in \mathbb{R}_{>0}. \tag{A4}$$

These assumptions guarantee that the system will only exhibit finite linearized strain.

Under the assumptions (A1)–(A4) stated above, there exists a positive constant C such that the following inequalities hold for all $\mathbf{R}_1, \mathbf{R}_2 \in \mathbb{R}^{d\times d}$:

$$(\mu(|\mathbf{R}_1|)\mathbf{R}_1 - \mu(|\mathbf{R}_2|)\mathbf{R}_2) : (\mathbf{R}_1 - \mathbf{R}_2) \geq C \frac{|\mathbf{R}_1 - \mathbf{R}_2|^2}{(\kappa + |\mathbf{R}_1| + |\mathbf{R}_2|)^{1+\alpha}};$$

$$(\mu(|\mathbf{R}_1|)\mathbf{R}_1 - \mu(|\mathbf{R}_2|)\mathbf{R}_2) : (\mathbf{R}_1 - \mathbf{R}_2) \geq C \left|(\kappa + |\mathbf{R}_1|)^{\frac{1-\alpha}{2}} - (\kappa + |\mathbf{R}_2|)^{\frac{1-\alpha}{2}}\right|^2;$$

$$(\lambda(\mathrm{tr}(\mathbf{R}_1))\mathrm{tr}(\mathbf{R}_1) - \lambda(\mathrm{tr}(\mathbf{R}_2))\mathrm{tr}(\mathbf{R}_2))\,(\mathrm{tr}(\mathbf{R}_1) - \mathrm{tr}(\mathbf{R}_2)) \geq 0.$$

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If, in addition,

$$0 < \frac{\mathrm{d}}{\mathrm{d}s}(\lambda(s)s) \qquad \forall\, s \in \mathbb{R}, \tag{A3'}$$

then, for all $\mathbf{R}_1, \mathbf{R}_2 \in \mathbb{R}^{d\times d}$ such that $\mathrm{tr}(\mathbf{R}_1) \neq \mathrm{tr}(\mathbf{R}_2)$, we have

$$(\lambda(\mathrm{tr}(\mathbf{R}_1))\mathrm{tr}(\mathbf{R}_1) - \lambda(\mathrm{tr}(\mathbf{R}_2))\mathrm{tr}(\mathbf{R}_2))\,(\mathrm{tr}(\mathbf{R}_1) - \mathrm{tr}(\mathbf{R}_2)) > 0.$$

The system (1), (2) is supplemented with the boundary conditions

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial_D\Omega \qquad \text{and} \qquad \mathbf{T}\boldsymbol{\nu} = \boldsymbol{\ell} \quad \text{on } \partial_N\Omega,$$

with $\partial_D\Omega$ and $\partial_N\Omega$, disjoint and $\overline{\partial_D\Omega \cup \partial_N\Omega} = \partial\Omega$, and $\boldsymbol{\nu}$ is the unit outward normal to $\partial\Omega$, $\mathbf{g} : \partial\Omega \to \mathbb{R}^d$ is a given displacement on $\partial_D\Omega$, and $\boldsymbol{\ell} : \partial\Omega \to \mathbb{R}^d$ is a given traction force on $\partial_N\Omega$.

Theorem (Theorem 4.3 in BMRS2014)

Assume $\partial_N \Omega = \emptyset$ *and* λ, μ *satisfy* (A1)–(A4) *with* $0 \le \alpha < 1/d$; *then:*

1. *Assume that* $\mathbf{f} = -\mathrm{div}(\mathbf{F})$ *for* $\mathbf{F} \in W^{\beta,1}(\Omega)^{d\times d}_{\mathrm{sym}}$ *with* $\beta \in (\alpha d, 1)$. *Then, there exists a pair* $(\mathbf{T}, \mathbf{u})$, *such that*

$$\mathbf{T} \in L_1(\Omega)^{d\times d}_{\mathrm{sym}}, \quad \mathbf{u} \in W^{1,p}_0(\Omega)^d, \quad p \in [1,\infty), \quad \varepsilon(\mathbf{u}) \in L_\infty(\Omega)^{d\times d}_{\mathrm{sym}}$$

is a weak solution in the sense that it satisfies

$$\int_\Omega \mathbf{T} : \varepsilon(\mathbf{w})\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{F} : \varepsilon(\mathbf{w})\,\mathrm{d}\mathbf{x} \qquad \forall\, \mathbf{w} \in \mathscr{D}(\Omega)^d, \tag{3}$$

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2. *Also,* $\mathbf{u}$ *is unique and if* λ *satisfies* (A3'), *then* $\mathbf{T}$ *is also unique;*
3. *Furthermore, if* $\mathbf{F}$ *belongs to* $W^{2,2}(\Omega)^{d\times d}_{\mathrm{sym}}$*, then* $\mathbf{T} \in W^{1,q}_{\mathrm{loc}}(\Omega)^{d\times d}_{\mathrm{sym}}$ *with* $q \in [1,2)$ *when* $d = 2$ *and* $q = 2 - \frac{1+\alpha}{2-\alpha}$ *when* $d = 3$.

Remark

When $\partial\Omega_N \neq \emptyset$ the structure of the solution is much more complicated.

In [BBMS2017] it was shown that the solution in that case belongs to the space of Radon measures, but if the problem is equipped with an asymptotic radial structure, then the solution can be understood as a standard weak solution, with one proviso: the attainment of the bdry value is penalized by a measure supported on $\partial\Omega_N$.

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The proof is based on constructing a sequence of solutions to a regularized problem, where the stress-strain relationship (2) is replaced by

$$\varepsilon(\mathbf{u}) = \lambda(\mathrm{tr}(\mathbf{T}))\mathrm{tr}(\mathbf{T})\mathbf{I} + \mu(|\mathbf{T}^{\mathbf{d}}|)\mathbf{T}^{\mathbf{d}} + \frac{\mathrm{tr}(\mathbf{T})\mathbf{I}}{n|\mathrm{tr}(\mathbf{T})|^{1-\frac{1}{n}}} + \frac{\mathbf{T}^{\mathbf{d}}}{n|\mathbf{T}^{\mathbf{d}}|^{1-\frac{1}{n}}};$$

here $n \in \mathbb{N}$ is a regularization parameter, and we shall let $n \to \infty$.

We shall consider the finite element approximation of this regularized problem, whose variational form is: find $(\mathbf{T}_n, \mathbf{u}_n) \in \mathbb{M}_n \times \mathbb{X}_n$ satisfying

$$a_n(\mathbf{T}_n, \mathbf{S}) + c(\mathbf{T}_n; \mathbf{T}_n, \mathbf{S}) - b(\mathbf{S}, \mathbf{u}_n) = 0 \qquad \forall\, \mathbf{S} \in \mathbb{M}_n,$$

$$b(\mathbf{T}_n, \mathbf{v}) = \int_\Omega \mathbf{F} : \varepsilon(\mathbf{v}) \,\mathrm{d}\mathbf{x} \qquad \forall\, \mathbf{v} \in \mathbb{X}_n,$$

where

$$a_n(\mathbf{T}, \mathbf{S}) := \frac{1}{n} \int_\Omega \left(\frac{\mathrm{tr}(\mathbf{T})\mathbf{I}}{|\mathrm{tr}(\mathbf{T})|^{1-\frac{1}{n}}} + \frac{\mathbf{T}^{\mathbf{d}}}{|\mathbf{T}^{\mathbf{d}}|^{1-\frac{1}{n}}} \right) : \mathbf{S} \,\mathrm{d}\mathbf{x},$$

$$c(\mathbf{T}; \mathbf{R}, \mathbf{S}) := \int_\Omega \Big(\lambda(\mathrm{tr}(\mathbf{T}))\mathrm{tr}(\mathbf{R})\mathbf{I} + \mu(|\mathbf{T}^{\mathbf{d}}|)\mathbf{R}^{\mathbf{d}} \Big) : \mathbf{S} \,\mathrm{d}\mathbf{x},$$

$$b(\mathbf{S}, \mathbf{v}) := \int_\Omega \mathbf{S} : \varepsilon(\mathbf{v}) \,\mathrm{d}\mathbf{x}$$

and

$$\mathbb{M}_n := L_{1+\frac{1}{n}}(\Omega)^{d\times d}_{\mathrm{sym}}, \qquad \mathbb{X}_n := W_0^{1,n+1}(\Omega)^d, \qquad n \in \mathbb{N}.$$

Existence/uniqueness of solution to the regularized problem

Lemma

For each $n \in \mathbb{N}$ *there exists a unique solution pair* $(\mathbf{T}_n, \mathbf{u}_n) \in \mathbb{M}_n \times \mathbb{X}_n$ *to the regularized problem.*

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Lemma (A-priori estimates)

Suppose that $\mathbf{F} \in L_{1+\frac{1}{n}}(\Omega)^{d\times d}_{\mathrm{sym}}$, and that λ and μ satisfy the properties (A1) *and* (A2). *Then,*

$$\|\varepsilon(\mathbf{u}_n)\|_{L_{n+1}(\Omega)} \leq C(d, |\Omega|^{\frac{1}{n}}) \left[\frac{1}{n}\|\mathbf{F}\|^{1+\frac{1}{n}}_{L_{1+\frac{1}{n}}(\Omega)} + \|\mathbf{F}\|_{L_{1+\frac{1}{n}}(\Omega)} + \kappa\right]^{\frac{1}{n+1}}$$

and

$$\frac{1}{n}\|\mathbf{T}_n\|^{1+\frac{1}{n}}_{L_{1+\frac{1}{n}}(\Omega)} + \|\mathbf{T}_n\|_{L_1(\Omega)} \leq C(d) \left[\frac{1}{n}\|\mathbf{F}\|^{1+\frac{1}{n}}_{L_{1+\frac{1}{n}}(\Omega)} + |\Omega|^{\frac{1}{n}}\|\mathbf{F}\|_{L_{1+\frac{1}{n}}(\Omega)} + \kappa|\Omega|\right].$$

Passage to the limit $n \to \infty$

$$\begin{aligned} \mathbf{u}_n &\rightharpoonup \mathbf{u} && \text{weakly in } W_0^{1,2d}(\Omega)^d, \\ \mathbf{u}_n &\to \mathbf{u} && \text{strongly in } \mathscr{C}(\overline{\Omega})^d, \\ \frac{(\mathbf{T}_n)^{\mathbf{d}}}{n|(\mathbf{T}_n)^{\mathbf{d}}|^{1-\frac{1}{n}}} &\to \mathbf{0} && \text{strongly in } L_1(\Omega)^{d\times d}_{\mathrm{sym}}, \\ \frac{\mathrm{tr}(\mathbf{T}_n)}{n|\mathrm{tr}(\mathbf{T}_n)|^{1-\frac{1}{n}}} &\to 0 && \text{strongly in } L_1(\Omega), \\ \varepsilon(\mathbf{u}_n) &\rightharpoonup \varepsilon(\mathbf{u}) && \text{weakly in } L_{2d}(\Omega)^{d\times d}_{\mathrm{sym}}, \end{aligned}$$

and

$$\varepsilon(\mathbf{u}_n) \to \varepsilon(\mathbf{u}) \qquad \text{strongly in } L_p(\Omega_0)^{d\times d} \qquad \forall\, \Omega_0 \subset\subset \Omega, \quad \forall\, p \in [1,\infty).$$

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Also, by a fractional Nikolskiĭ norm bound and embedding [BMRS2014]:

$$\mathbf{T}_n \to \mathbf{T} \text{ in } L_q(\Omega_0)^{d\times d}_{\mathrm{sym}}, \qquad \forall\, q \in \left[1, 1 + \tfrac{1}{2}\tfrac{\beta-\alpha d}{d-\beta}\right), \quad \begin{cases} \beta \in (\alpha d, 1), \\ 0 \le \alpha < \frac{1}{d}, \end{cases} \quad \Omega_0 \subset\subset \Omega.$$

Finite element approximation

Suppose Ω is a polygon when $d = 2$ or a Lipschitz polyhedron when $d = 3$.

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Consider a sequence of shape-regular simplicial subdivisions $(\mathscr{T}_h)_{h\in(0,1]}$ of $\overline{\Omega}$; i.e. there exists a positive real number η, independent of the mesh-size h, such that all closed simplices K contained in $\mathscr{T}_h$ satisfy

$$\frac{h_K}{\varrho_K} \leq \eta,$$

where $h_K := \text{diam}(K)$ and ϱ_K is the diameter of the largest ball $\subset K$.

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Let $\mathscr{P}_h^r$ be the space of piecewise (subordinate to $\mathscr{T}_h$) polynomials of degree at most r. We consider the conforming finite element spaces

$$\mathbb{M}_{n,h} := \left(\mathscr{P}_h^0\right)^{d\times d}_{\mathrm{sym}} \subset \mathbb{M}_n, \qquad \mathbb{X}_{n,h} := \left(\mathscr{P}_h^1\right)^d \cap \mathbb{X}_n \subset \mathbb{X}_n,$$

for the approximation of $\mathbf{T}_n$ and $\mathbf{u}_n$, respectively.

Discrete scheme

The discrete counterpart of the regularized problem, based on $\mathbb{X}_{n,h}$ and $\mathbb{M}_{n,h}$, is defined as follows: find $(\mathbf{T}_{n,h}, \mathbf{u}_{n,h}) \in \mathbb{M}_{n,h} \times \mathbb{X}_{n,h}$ such that

$$
\begin{aligned}
&a_n(\mathbf{T}_{n,h}, \mathbf{S}_h) + c(\mathbf{T}_{n,h}; \mathbf{T}_{n,h}, \mathbf{S}_h) - b(\mathbf{S}_h, \mathbf{u}_{n,h}) = 0 \qquad && \forall\, \mathbf{S}_h \in \mathbb{M}_{n,h}, \\
&b(\mathbf{T}_{n,h}, \mathbf{v}_h) = \int_\Omega \mathbf{F} : \varepsilon(\mathbf{v}_h)\, \mathrm{d}\mathbf{x} && \forall\, \mathbf{v}_h \in \mathbb{X}_{n,h}.
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Lemma (Weak convergence of $\mathbf{T}_{n,h}$)

Assume that $\mathbf{F} \in L_{1+\frac{1}{n}}(\Omega)^{d\times d}_{\mathrm{sym}}$ *and that the functions* λ *and* μ *satisfy the hypotheses* (A1)–(A4). *Let* $(\mathbf{T}_n, \mathbf{u}_n) \in \mathbb{M}_n \times \mathbb{X}_n$ *be the unique solution of the regularized problem. Then, as* $h \to 0_+$,

$$
\mathbf{T}_{n,h} \rightharpoonup \mathbf{T}_n \qquad \textit{weakly in } \mathbb{M}_n = L_{1+\frac{1}{n}}(\Omega)^{d\times d}_{sym}.
$$

Lemma (Strong convergence)

Let $\mathbf{F} \in L_{1+\frac{1}{n}}(\Omega)^{d\times d}_{\mathrm{sym}}$, *and assume that* λ *and* μ *satisfy the assumptions* (A1)–(A4). *Let* $(\mathbf{T}_n, \mathbf{u}_n)$ *denote the unique solution to the regularized problem, with* $n \in \mathbb{N}$. *Then, for each fixed* $n \in \mathbb{N}$, *as* $h \to 0_+$,

$$\begin{aligned} \mathbf{T}_{n,h} &\to \mathbf{T}_n && \textit{strongly in } L_p(\Omega)^{d\times d}_{sym} \textit{ for all } p \in \big[1, 1+\tfrac{1}{n}\big), \\ \varepsilon(\mathbf{u}_{n,h}) &\rightharpoonup \varepsilon(\mathbf{u}_n) && \textit{weakly in } L_p(\Omega)^{d\times d}_{\mathrm{sym}} \textit{ for all } p \in [1, n+1]. \end{aligned}$$

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Furthermore, if λ *satisfies* (A3'), *we have that, for any* $\Omega_0 \subset\subset \Omega$,

$$\lim_{n\to\infty} \lim_{h\to 0_+} \|\mathbf{T}_{n,h} - \mathbf{T}\|_{L_1(\Omega_0)} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \lim_{h\to 0_+} \|\mathbf{u}_{n,h} - \mathbf{u}\|_{C(\overline{\Omega})} = 0,$$

$$\lim_{n\to\infty} \lim_{h\to 0_+} \|\varepsilon(\mathbf{u}_{n,h}) - \varepsilon(\mathbf{u})\|_{L_p(\Omega_0)} = 0 \qquad \forall\, \Omega_0 \subset\subset \Omega, \quad \forall p \in [1, \infty).$$

Theorem

In addition to the assumptions of the previous lemma also suppose that the functions $s \in \mathbb{R} \mapsto \lambda(s)s \in \mathbb{R}$ *and* $\mathbf{S} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \mapsto \mu(|\mathbf{S}|)\mathbf{S} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ *are Hölder-continuous with exponent* $\beta \in (0,1]$*, i.e., there exists a* $\Lambda > 0$ *s.t.*

$$|\lambda(r)r - \lambda(s)s| \le \Lambda |r-s|^\beta \quad \forall\, r,s \in \mathbb{R},$$

$$|\mu(|\mathbf{R}|)\mathbf{R} - \mu(|\mathbf{S}|)\mathbf{S}| \le \Lambda |\mathbf{R}-\mathbf{S}|^\beta \quad \forall\, \mathbf{R},\mathbf{S} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}.$$

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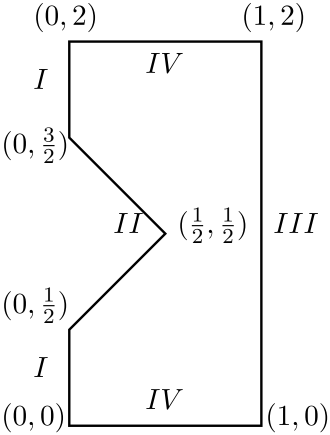
Then, for any $n \in \mathbb{N}$*,* $\zeta \in (0,1]$ *and* $\gamma \in (0,1]$*, as* $h \to 0_+$*,*

$$\|\mathbf{T}_{n,h} - \Pi_h \mathbf{T}_n\|_{L_1(\Omega)} \leq C h^{\frac{\gamma}{n}} \left(\|\varepsilon(\mathbf{u}_n)\|_{W^{\frac{\gamma}{n},\infty}(\Omega)} + \|\mathbf{T}_n\|^{\frac{1}{n}}_{W^{\gamma,\infty}(\Omega)} \right),$$

as $h \to 0_+$*, where* $C = C(d, \Lambda, n, \gamma, |\Omega|)$*.*

Crack problem



A horizontal compressive force $\mathbf{T}\boldsymbol{\nu} = (f, 0)^{\mathrm{T}}$ for $f > 0$ is applied on the side III, while no force (i.e., $\mathbf{T}\boldsymbol{\nu} = \mathbf{0}$) is imposed on the side marked by I and II. The top and bottom sides are fixed, i.e., $\mathbf{u} = \mathbf{0}$.

Set $\lambda(s) = \mu(s) = (1+s^2)^{-\frac{1}{2}}$, $n = 100$. The domain is partitioned into 16384 elements of minimal diameter $h = 0.011$.

The figures show the deformed domain, for different force-magnitudes $f = 0.25, 0.5, 0.75, 1$ (from left to right) pulling the right edge, edge III, of the original domain. The grey scale indicates the magnitude of the displacement $|\mathbf{u}|$, where white corresponds to 0 and black to 0.92.

The table below reports the variation of $\|\nabla \mathbf{u}_h\|_{L_\infty(\Omega)}$ and $\|\mathbf{T}_h\|_{L_\infty(\Omega)}$ as the magnitude of the force increases. The influence of the latter is severe on $\|\mathbf{T}_h\|_{L_\infty(\Omega)}$ while relatively moderate on $\|\varepsilon(\mathbf{u}_h)\|_{L_\infty(\Omega)} \leq \|\nabla \mathbf{u}_h\|_{L_\infty(\Omega)}$.

	$f = 0.25$	$f = 0.5$	$f = 0.75$	$f = 1$	$f = 1.25$	$f = 1.5$
$\|\nabla \mathbf{u}_{n,1,h}\|_{L_\infty(\Omega)}$	1.0656	2.2510	3.5032	5.2703	7.0492	8.8003
$\|\mathbf{T}_{n,1,h}\|_{L_\infty(\Omega)}$	0.92231	5.3090	18.17	46.5215	95.3902	166.335

This is in accordance with the properties of the strain-limiting model.

Open questions and extensions

- Optimal error bounds for $\mathbf{T}$ as $h \to 0$? An idea may be to use that:

$$\frac{|\nabla \mathbf{T}_n|^2}{(1+|\mathbf{T}_n|^a)^{1+\frac{1}{a}}} \overset{b}{\in} L^1_{\mathrm{loc}}(\Omega), \qquad a > 0.$$

In addition:

$$\log(1+|\mathbf{T}_n|) \overset{b}{\in} W^{1,2}_{\mathrm{loc}}(\Omega) \qquad \text{for } d = 2,$$

$$(1+|\mathbf{T}_n|^2)^{\frac{2-q}{2}} \overset{b}{\in} W^{1,2}_{\mathrm{loc}}(\Omega) \qquad \text{for } d = 3 \text{ and } 1 \le q < \tfrac{5}{3}.$$

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- Convergence analysis in the case of mixed Dirichlet–Neumann boundary conditions?

M Bulíček, V Patel, Y. Şengül, and E Süli. Existence of large-data global weak solutions to a model of a strain-limiting viscoelastic body. Comm. on Pure and Applied Analysis. May 2021, 20(5): 1931-1960.

M Bulíček, V Patel, Y. Şengül, and E Süli. Existence and uniqueness of global weak solutions to strain-limiting viscoelasticity with Dirichlet boundary data. (Submitted). Available from: arXiv:2011.11683 [math.AP].