Splitting methods in Approximate Bayesian Computation for partially observed diffusion processes

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Outline

1. Jansen and Rit Neural Mass Model (JR-NMM)

2. Structure-Preserving Approximate Bayesian Computation (ABC)

3. ABC Results on Simulated Data and Real EEG Data Application

Jansen and Rit Neural Mass Model (JR-NMM)

Neural Mass Models

- Provide a mathematical framework for studying neural brain activity
- Model whole populations of neurons with average properties
- Reproduce EEG (Electroencephalography) rhythms
- Applied in neurological disorders (epilepsy, schizophrenia, etc.)

Structure-Preserving ABC for the JR-NMM from EEG Data

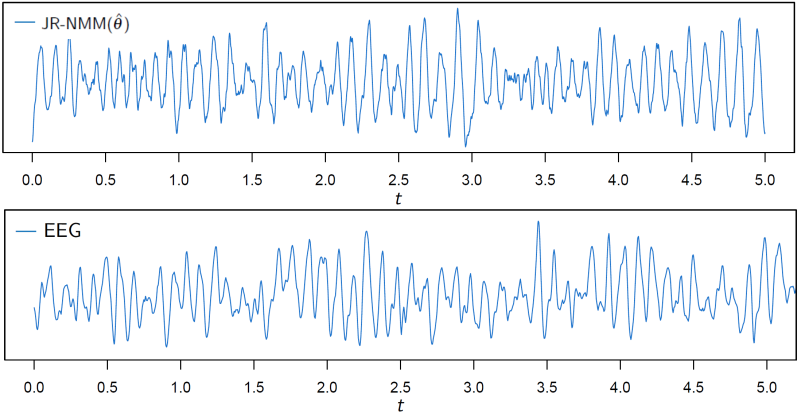

Figure: Sample path for $\hat{\theta}$ from the JR-NMM versus an EEG segment

The Original Jansen and Rit Neural Mass Model (JR-NMM)[1]

Original JR-NMM

$$\dot{x}_0(t) = x_3(t) \qquad \text{Pyramidal cells (Main population)}$$

$$\dot{x}_1(t) = x_4(t) \qquad \text{Excitatory interneurons (Subpopulation 1)}$$

$$\dot{x}_2(t) = x_5(t) \qquad \text{Inhibitory interneurons (Subpopulation 2)}$$

$$\dot{x}_3(t) = Aa \cdot \mathrm{Sigm}(x_1(t) - x_2(t)) - 2a \cdot x_3(t) - a^2 \cdot x_0(t)$$

$$\dot{x}_4(t) = Aa \cdot [p(t) + \frac{4}{5} C \cdot \mathrm{Sigm}(C \cdot x_0(t))] - 2a \cdot x_4(t) - a^2 \cdot x_1(t)$$

$$\dot{x}_5(t) = \frac{1}{4} C Bb \cdot \mathrm{Sigm}(\frac{1}{4} C \cdot x_0(t)) - 2b \cdot x_5(t) - b^2 \cdot x_2(t)$$

- Related to the **EEG**-signal: $y(t) = x_1(t) - x_2(t)$
- 8 biologically motivated **parameters**: $A, B, a, b, \nu_{max}, r, v_0, C$
- **Nonlinear** sigmoid function: $\mathrm{Sigm}(x) := \frac{\nu_{max}}{1+e^{r(v_0-x)}}$
- Excitatory (**stochastic**) **input** from neighbouring columns: $p(t)$

[1] B.H. Jansen and V.G. Rit. "Electroencephalogram and visual evoked potential generation in a mathematical model of coupled cortical columns." In: Biological cybernetics, 73(4):357-366 (1995)

How can we study the model/problem?

- **Mathematical level**
  Dynamical and structural properties
- **Numerical level**
  Efficient and structure-preserving simulation
- **Statistical level**
  Parameter estimation

Reformulate the *ODE with random input* as an **SDE**

The Stochastic Jansen and Rit Neural Mass Model[2]

Stochastic JR-NMM

$$
\begin{aligned}
dX_0(t) &= X_3(t)\; dt \\
dX_1(t) &= X_4(t)\; dt \\
dX_2(t) &= X_5(t)\; dt \\
dX_3(t) &= \{Aa \cdot \mathrm{Sigm}(X_1(t) - X_2(t)) - 2a \cdot X_3(t) - a^2 \cdot X_0(t)\}\; dt + \sigma_3\; dW_3(t) \\
dX_4(t) &= \{Aa \cdot [\mu + \frac{4}{5} C \cdot \mathrm{Sigm}(C \cdot X_0(t))] - 2a \cdot X_4(t) - a^2 \cdot X_1(t)\}\; dt + \sigma_4\; dW_4(t) \\
dX_5(t) &= \{\frac{1}{4} C Bb \mathrm{Sigm}(\frac{1}{4} C \cdot X_0(t)) - 2b \cdot X_5(t) - b^2 \cdot X_2(t)\}\; dt + \sigma_5\; dW_5(t)
\end{aligned}
$$

- $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions
- Independent and $\mathcal{F}_t$-adapted Wiener processes: $W_i(t)$, $i = 3, 4, 5$
- Diffusion components: $\sigma_i > 0$, $i = 3, 4, 5$
- Globally Lipschitz drift and diffusion imply existence of a unique $\mathcal{F}_t$-adapted strong solution

[2] M. Ableidinger, E. Buckwar, and H. Hinterleitner. "A Stochastic Version of the Jansen and Rit Neural Mass Model: Analysis and Numerics." In: Journal of Mathematical Neuroscience 7(8) (2017)

The Stochastic Jansen and Rit Neural Mass Model

Stochastic JR-NMM

$$dX_0(t) = X_3(t)\ dt$$
$$dX_1(t) = X_4(t)\ dt$$
$$dX_2(t) = X_5(t)\ dt$$
$$dX_3(t) = \{Aa \cdot \mathrm{Sigm}(X_1(t) - X_2(t)) - 2a \cdot X_3(t) - a^2 \cdot X_0(t)\}\ dt + \sigma_3\ dW_3(t)$$
$$dX_4(t) = \{Aa \cdot [\mu + \frac{4}{5} C \cdot \mathrm{Sigm}(C \cdot X_0(t))] - 2a \cdot X_4(t) - a^2 \cdot X_1(t)\}\ dt + \sigma_4\ dW_4(t)$$
$$dX_5(t) = \{\frac{1}{4} C Bb \mathrm{Sigm}(\frac{1}{4} C \cdot X_0(t)) - 2b \cdot X_5(t) - b^2 \cdot X_2(t)\}\ dt + \sigma_5\ dW_5(t)$$

Parameters of interest: $\theta = (C, \mu)$

1. Parameter C (α-rhythmic value, original JR-NMM literature: 135)
   Internal connectivity

2. Parameter μ (not studied in the literature)
   Deterministic external input

Setting for Parameter Inference

1. *The model*: 6-dim **solution process** of the stochastic JR-NMM

$$\mathbf{X} = (X_0(t), ..., X_5(t))^T, \quad t \in [0, T]$$

2. *Available data*: **EEG**-related 1-dim **output process**

$$Y_\theta(t) = X_1(t) - X_2(t), \quad t \in [0, T]$$

$\Longrightarrow$ Inference for partially observed stochastic processes

3. *Goal*: Statistical inference on the parameters

$$\theta = (C, \mu)$$

from **EEG** time series **data** y

4. *Statistical issue*: intractable likelihood
$\Longrightarrow$ Likelihood-free inference

Structure-Preserving Approximate Bayesian Computation (ABC)

(Likelihood-free) Approximate Bayesian Computation - ABC[3]

Bayesian Computation

$$\underbrace{\pi(\theta|y)}_{posterior} \propto \underbrace{\pi(y|\theta)}_{\substack{likelihood \\ (intractable)}} \underbrace{\pi(\theta)}_{prior}$$

Approximate Bayesian Computation

$$\pi(\theta|y) \approx \pi_{d,\epsilon,s}(\theta|y) = \pi(\ \theta \mid d(s(y), s(y_\theta)) < \epsilon\ )$$

Structure-preserving ABC for the stochastic JR-NMM

$$\pi(\theta|y) \approx \pi_{d,\epsilon,s,\hat{y}_\theta}(\theta|y) = \pi(\ \theta \mid d(s(y), s(\hat{y}_\theta)) < \epsilon\ )$$

1. New interpretation of s: Incorporate SDE dynamics and structure
2. Reliable numerics to simulate $\hat{y}_\theta$: Efficient and structure-preserving

[3] M.A. Beaumont, W. Zhang, D.J. Balding "Approximate Bayesian computation in population genetics." Genetics, 162(4):2025-2035 (2002)

JR-NMM: Hamiltonian Stochastic Differential Equation

$$dQ(t) = \nabla_P H(Q,P)dt$$
$$dP(t) = [\underbrace{-\nabla_Q H(Q,P)}_{\text{Hamiltonian}} - \underbrace{2\Gamma P}_{\text{linear damping}} + \underbrace{G(\theta, Q)}_{\text{nonlinear displacement}}]\, dt + \underbrace{\Sigma}_{\text{diffusion matrix}} dW(t)$$

- $H : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ with $H(Q,P) := \frac{1}{2}(\|P\|^2_{\mathbb{R}^3} + \|\Gamma Q\|^2_{\mathbb{R}^3})$
- $Q = (X_0, X_1, X_2)^T \in \mathbb{R}^3,\ P = (X_3, X_4, X_5)^T \in \mathbb{R}^3$
- $G(\theta, Q) = \begin{pmatrix} Aa \cdot \text{Sigm}(X_1 - X_2) \\ Aa \cdot [\mu + \frac{4}{5} C \cdot \text{Sigm}(C \cdot X_0)] \\ \frac{1}{4} C Bb \cdot \text{Sigm}(\frac{1}{4} C \cdot X_0) \end{pmatrix} \in \mathbb{R}^3$

Theorem (Structural property - Ergodicity)

Let $Y_\theta = (Y_\theta(t))_{t \geq 0}$ *be the* output process *of the stochastic JR-NMM.*

1. *The process* Y_θ *has a unique invariant meausure* η_{Y_θ} *on* $\mathbb{R}$*.*
2. Y_θ *converges exponentially fast towards its stationary regime* η_{Y_θ}*.*

Choice of the Numerical Data Generation Scheme $\hat{y}_\theta$ in ABC

$$\pi(\theta|y) \approx \pi_{d,\epsilon,s,\hat{y}_\theta}(\theta|y) = \pi(\ \theta \mid d(s(y), s(\hat{y}_\theta)) < \epsilon\ )$$

Challenge:

$$\begin{aligned} dQ(t) &= \nabla_P H(Q,P)dt \\ \underbrace{dP(t) = [-\nabla_Q H(Q,P)}_{\text{Hamiltonian}} &- \underbrace{2\Gamma P}_{\substack{\text{linear}\\ \text{damping}}} + \underbrace{G(\theta,Q)}_{\substack{\text{nonlinear}\\ \text{displacement}}}]\,dt + \underbrace{\Sigma}_{\substack{\text{diffusion}\\ \text{matrix}}} dW(t) \end{aligned}$$

$\Longrightarrow$ No exact simulation scheme available

Approach:

$\Longrightarrow$ Numerical splitting (Efficient and preserves the structure η_{Y_θ})

A little detour: The idea of splitting methods

Illustration with simple deterministic ODEs

The very basic idea of splitting methods

Consider deterministic ODEs

$$\dot{y} = f(y), \quad y(0) = y_0\,, t \in [0, T]\,. \tag{1}$$

Step 1 consists of rearranging the function f as a sum

$$f(y) = f^{[1]}(y) + f^{[2]}(y)\,,$$

to obtain subequations

$$\dot{y}^{[1]} = f^{[1]}(y^{[1]}) \quad \text{and} \quad \dot{y}^{[2]} = f^{[2]}(y^{[2]})\,.$$

Step 2 consists of solving each of the subequations analytically, yielding the explicit flows

$$\varphi_t^{[1]}(y_0^{[1]}) \quad \text{and} \quad \varphi_t^{[2]}(y_0^{[2]})$$

Step 3 consists of choosing how to compose the flows to obtain a numerical method producing the iterates $y_n \approx y(t_n)$ along the grid on $[0, T]$ for the original equation (1):

- the Lie-Trotter composition yields

$$y_{n+1} = \left(\varphi_{\Delta t}^{[1]} \circ \varphi_{\Delta t}^{[2]}\right)(y_n)$$

- the Strang composition yields

$$y_{n+1} = \left(\varphi_{\Delta t/2}^{[1]} \circ \varphi_{\Delta}^{[2]} \circ \varphi_{\Delta t/2}^{[1]}\right)(y_n)$$

Most simple example ever

Consider a linear system of ODEs

$$\dot{y} = Ay, \quad y(0) = y_0\,, t \in [0, T] \qquad \text{with solution} \quad y(t) = \exp(At)y_0\,. \tag{2}$$

Step 1 consists of rewriting the matrix A as $A = A^{[1]} + A^{[2]}$ to obtain subequations

$$\dot{y}^{[1]} = A^{[1]}y^{[1]} \quad \text{and} \quad \dot{y}^{[2]} = A^{[2]}y^{[2]}\,.$$

Step 2 consists of solving each of the subequations analytically, yielding the explicit flows

$$\varphi_t^{[1]}(y_0^{[1]}) = \exp(A^{[1]}t)y_0^{[1]} \quad \text{and} \quad \varphi_t^{[2]}(y_0^{[2]}) = \exp(A^{[2]}t)y_0^{[2]}$$

Step 3 consists of choosing how to compose the flows to obtain a numerical method producing the iterates $y_n \approx y(t_n)$ along the grid on $[0, T]$ for the original equation (1):

- the Lie-Trotter composition yields

$$y_{n+1} = \left(\varphi_{\Delta t}^{[1]} \circ \varphi_{\Delta t}^{[2]}\right)(y_n) = \exp(A^{[1]}\Delta t)\cdot\exp(A^{[2]}\Delta t)\, y_n = \exp(A^{[1]}\Delta t + A^{[2]}\Delta t)y_n\,,$$

- the Strang composition yields

$$y_{n+1} = \left(\varphi_{\Delta t/2}^{[1]} \circ \varphi_{\Delta}^{[2]} \circ \varphi_{\Delta t/2}^{[1]}\right)(y_n) = \exp(A^{[1]}\Delta t/2)\cdot\exp(A^{[2]}\Delta t)\cdot\exp(A^{[1]}\Delta t/2)\, y_n$$
$$= \exp(A^{[1]}\Delta t/2 + A^{[2]}\Delta t + A^{[1]}\Delta t/2)y_n\,,$$

Preserving structure, Literature

Splitting methods are one way to go in Geometric Numerical Integration of ODEs. Properties/structure/geometries to be preserved include

- Invariants, such as first integrals,
- Hamiltonian structures
- Symplecticity
- Energy bounds
- Geometric structures

Small sample of literature:

- E.Hairer, C.Lubich, G.Wanner: Geometric Numerical Integration
- J. M. Sanz-Serna & M. P. Calvo: Numerical Hamiltonian Problems
- R. Glowinski, S.Osher, W.Yin (Eds.): Splitting Methods in Communication, Imaging, Science, and Engineering, 2016
- R.Quispel, R.McLachlan, Splitting Methods, Acta Numerica, 2002

Splitting in the Stochastic case

Preserving structure, Literature on Splitting in the stochastic case

Properties/structure/geometries to be preserved include

- Invariants, such as first integrals,
- Hamiltonian structures
- Symplecticity
- Energy bounds, usually in expectation
- Geometric structures
- Ergodicity
- Invariant measures

Early references:
T. Misawa: A Lie algebraic approach to numerical integration of stochastic differential equations. SIAM J. Sci. Comput., 23(3):866-890, 2001.
T. Misawa: Numerical integration of stochastic differential equations by composition methods. RIMS Kokyuroku 1180, 166-190 (2000).

Other authors include: T. Shardlow, B. Leimkuhler, M. Tretyakov, G. Milstein,

D. Cohen, J. Hong, T. Lyons, N. Victoir, C.E. Brehier, L. Goudenege, T. Yamada . . .

Splitting in the stochastic case

Consider the n-dimensional SDE system with an m-dimensional Wiener process

$$dY(t) = F(t, Y(t))\, dt + G(t, Y(t))\, dW(t), \quad Y(0) = y_0, \quad t \in [0, T],$$

Step 1 consists of rearranging the drift and diffusion functions into sums

$$F(t, Y(t)) = \sum_{l=1}^{d} F^{[l]}(t, Y(t)), \quad G(t, Y(t)) = \sum_{l=1}^{d} G^{[l]}(t, Y(t)), \quad d \in \mathbb{N},$$

to obtain subequations

$$dY^{[l]}(t) = F^{[l]}(t, Y^{[l]}(t))\, dt + G^{[l]}(t, Y^{[l]}(t))\, dW(t), \quad l \in \{1, ..., d\},$$

Step 2 consists of analytically solving the subequations to obtain explicit flows.

Step 3 consists of choosing composition schemes of the flows, as before.

- Note that for multiplicative noise Itô SDEs, it may be convenient to transform to the Stratonovich version.

Examples of splittings for Step 1 and 2 for SDEs

The Kubo oscillator, a 2-dim. Stratonovich SDE system with a 1-dim. Wiener process $(W(t))_{t\geq 0}$, parameters α, $\sigma > 0$ of the form

$$d\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}}_{=:A} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} dt + \underbrace{\begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}}_{=:\Sigma} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \circ dW(t), \quad \begin{pmatrix} X(0) \\ Y(0) \end{pmatrix} = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}.$$

Step 1: Setting $Z(t) = (X(t), Y(t))^\intercal$, subsystems are

$$dZ^{[1]}(t) = AZ^{[1]}(t)dt \quad \text{and} \quad dZ^{[2]}(t) = \Sigma Z^{[2]}(t) \circ dW(t).$$

Step 2 consists of solving each of the subequations analytically, yielding the explicit flows

$$\varphi_t^{[1]}(Z[1]_0) = \exp(At)Z_0^{[1]} \quad \text{and} \quad \varphi_t^{[2]}(Z_0^{[2]}) = \exp(\Sigma W(t))Z_0^{[2]}$$

We have

$$\exp(At) = \begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) \\ \sin(\alpha t) & \cos(\alpha t) \end{pmatrix} \exp(\Sigma W(t)) = \begin{pmatrix} \cos(\sigma W(t)) & -\sin(\sigma W(t)) \\ \sin(\sigma W(t)) & \cos(\sigma W(t)) \end{pmatrix}$$

$$\exp(At + \Sigma W(t)) = \begin{pmatrix} \cos(\alpha t + \sigma W(t)) & -\sin(\alpha t + \sigma W(t)) \\ \sin(\alpha t + \sigma W(t)) & \cos(\alpha t + \sigma W(t)) \end{pmatrix}$$

The matrices A and Σ commute, so the solution of the Kubo oscillator is $Z(t) = \exp(At + \Sigma W(t))Z_0$, also the splitting methods will compute the exact solution, thus the Kubo oscillator corresponds to the *most simple example ever* in the stochastic case.

Back to: Structure-Preserving Approximate Bayesian Computation (ABC)

Efficient and Structure-Preserving Numerical Splitting

$$\underbrace{\begin{array}{l} dQ(t) = \nabla_P H(Q,P)dt \\ dP(t) = [-\nabla_Q H(Q,P) \end{array}}_{\text{Hamiltonian}} - \underbrace{2\Gamma P}_{\substack{\text{linear}\\\text{damping}}} + \underbrace{G(\theta, Q)}_{\substack{\text{nonlinear}\\\text{displacement}}}]\, dt + \underbrace{\Sigma}_{\substack{\text{diffusion}\\\text{matrix}}} dW(t)$$

1. Subsystem a: linear SDE

$$\begin{pmatrix} dQ \\ dP \end{pmatrix} = \begin{pmatrix} \nabla_P H(Q,P) \\ -\nabla_Q H(Q,P) - 2\Gamma P \end{pmatrix} dt + \begin{pmatrix} 0_3 \\ \Sigma dW(t) \end{pmatrix}$$

2. Subsystem b: non-linear ODE

$$\begin{pmatrix} dQ \\ dP \end{pmatrix} = \begin{pmatrix} 0_3 \\ G(\theta, Q) \end{pmatrix} dt$$

Strang splitting in a discretized regime with equidistant time steps Δt:

$$\hat{\mathbf{X}} = X^b_{\Delta t/2} \circ X^a_{\Delta t} \circ X^b_{\Delta t/2}, \quad \hat{\mathbf{Y}}_\theta = \hat{X}_1 - \hat{X}_2$$

Efficient and Structure-Preserving Numerical Splitting

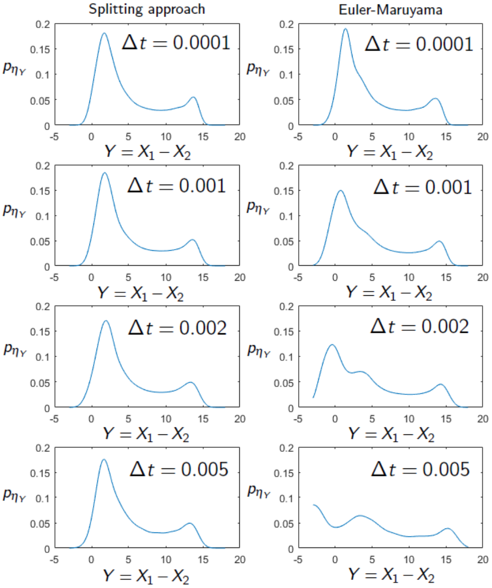

Figure: Invariant densities for $\theta = (C = 150, \mu = 220)$ and different Δt

A new Interpretation of Summaries s in ABC

$$\pi(\theta|y) \approx \pi_{d,\epsilon,s,\hat{y}_\theta}(\theta|y) = \pi(\ \theta \mid d(s(y), s(\hat{y}_\theta)) < \epsilon\ )$$

Challenge:

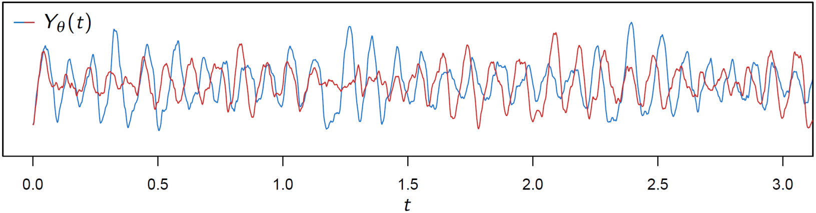

Figure: 2 sample paths for $\theta = (C = 135, \mu = 220)$

$\Longrightarrow$ How to account for the variability in the data for identical θ?

Approach:

$\Longrightarrow$ Transform the data from time to frequency domain

Spectral Density (Periodogram)

- Stationary stochastic process: $\mathbf{Y}_\theta = (Y_\theta(t))_{t\geq 0}$
- Autocovariance function: $\text{Cov}(Y_\theta(t), Y_\theta(s)) = r_\theta(\tau = t - s)$

$$S_{\mathbf{Y}_\theta}(\omega = 2\pi f) = \int_{-\infty}^{\infty} r_\theta(\tau) e^{-i\omega\tau} \, d\tau, \quad \omega \in [-\pi, \pi]$$

Definition (Periodogram)

$$s(y) = \hat{S}_y(\omega) = \frac{1}{J} \left| \sum_{j=1}^{J} y_j e^{-i\omega j} \right|^2$$

- Time domain: Discrete data $y = (y_1, ..., y_J)$, $J \in \mathbb{N}$
- Frequency domain: *R*-function `spectrum` (Fast Fourier Transform)

Simulated Data and Periodogram Estimates

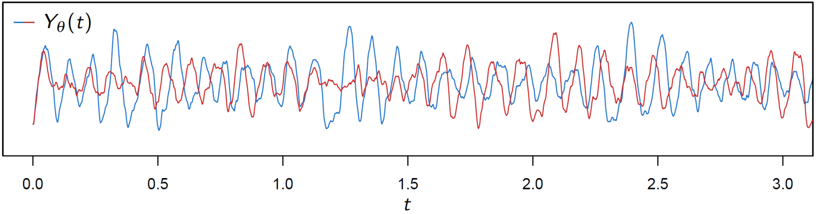

Figure: 2 sample paths for $\theta = (C = 135, \mu = 220)$

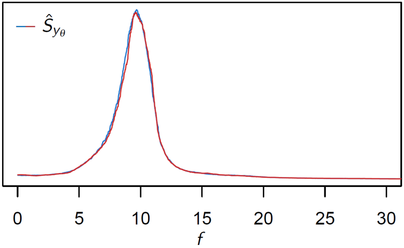

Figure: 2 (smoothed) periodograms for $\theta = (C = 135, \mu = 220)$

The Splitting Scheme Preserves the Periodogram

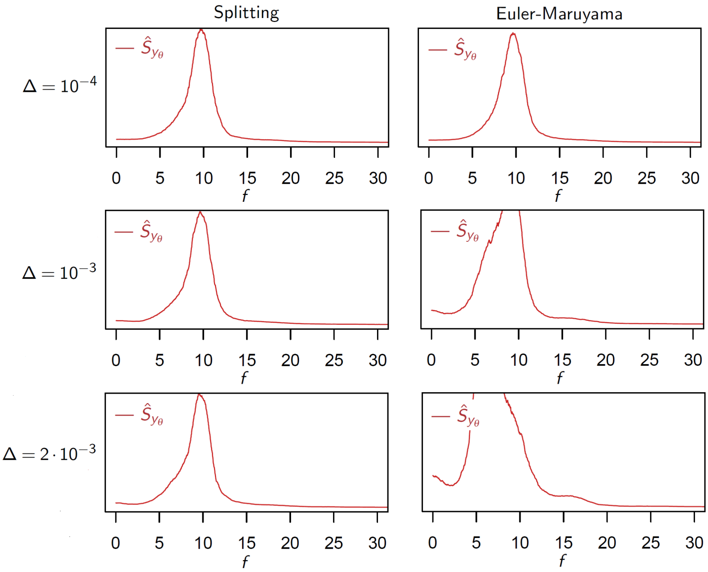

Figure: Periodogram estimates for $\theta = (C = 150, \mu = 220)$ and different Δt

Structure-Preserving ABC-Algorithm

ABC for the stochastic JR-NMM

Input: Observed datasets (EEG segments) $y = (y_1, ..., y_M)$, $M \in \mathbb{N}$

- Precompute the periodograms $s(y) = (\hat{S}_{y_1}, ..., \hat{S}_{y_M}) = (s_1, ..., s_M)$
- Choose prior distribution $\pi(\theta)$ and tolerance ϵ

for i=1:N do *Parallel simulation on the HPC (High Performance Cluster) Radon1*

- Draw θ_i from the prior $\pi(\theta)$
- Simulate new data $\hat{y}_{\theta_i}$ using the numerical splitting approach
- Compute the periodogram $s(\hat{y}_{\theta_i}) = \hat{s}_{\theta_i}$
- Calculate $D_i(s(y), \hat{s}_{\theta_i}) = \text{median}(\ d(s_1, \hat{s}_{\theta_i}), ..., d(s_M, \hat{s}_{\theta_i})\ )$
- Store samples (D_i, θ_i)

end for

If $D_i < \epsilon$, keep θ_i as a sample of the posterior $\pi_{d,\epsilon,s,\hat{y}_\theta}(\theta|y)$

Output: Samples $\theta_1, ..., \theta_n$ from the approximated posterior $\pi_{d,\epsilon,s,\hat{y}_\theta}(\theta|y)$

ABC Results on Simulated Data and Real EEG Data Application

ABC for the JR-NMM: Simulated Data

α-rhythmic behaviour:

- Original JR-NMM literature: $C = 135$
- Newly introduced in the SDE-version (not yet investigated): μ

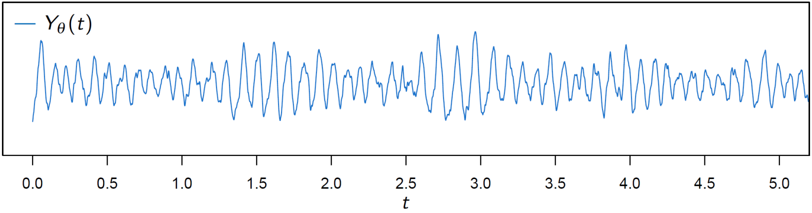

Figure: Sample path for $\theta = (C = 135, \mu = 220)$

Structure-Preserving ABC: Results for Simulated Data

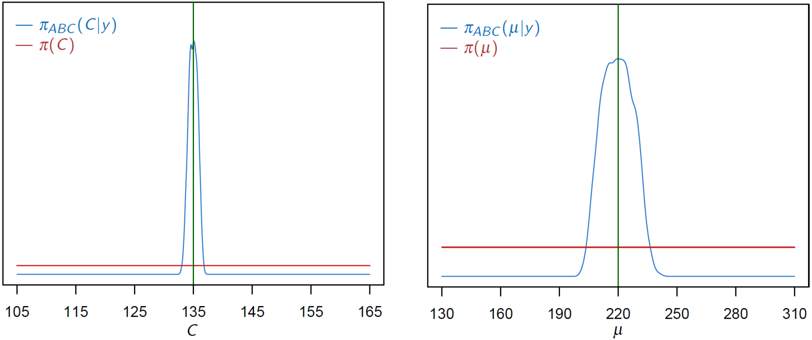

Figure: Marginal ABC posterior densities

- Prior distributions: $\pi(C) = U(105, 165)$, $\pi(\mu) = U(130, 310)$
- Number of simulated synthetic datasets: $N = 10^6$
- Data generation: time step $\Delta t = 2 \cdot 10^{-3}$, time horizon $T = 200$
- Tolerance: $\epsilon = 0.5^{th}$ percentile
- Observed datasets (independent repeated experiments): $M = 30$

Structure-Preserving ABC: Results for Simulated Data

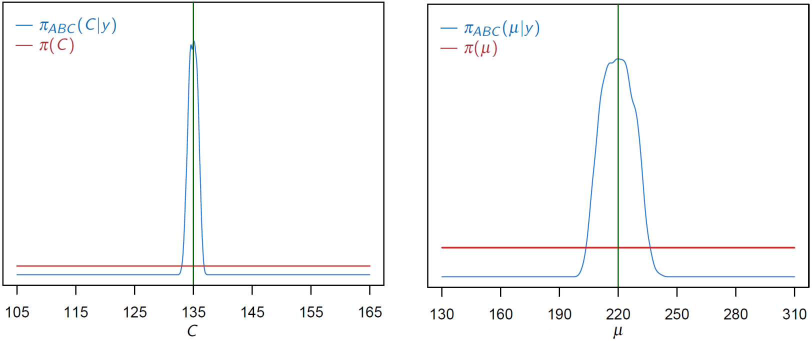

Figure: Marginal ABC posterior densities

	C	μ
ABC Posterior Mean	$\int_D C \cdot \pi_{ABC}(C|y)\ dC = 134.966 = \hat{C}$	$\int_D \mu \cdot \pi_{ABC}(\mu|y)\ d\mu = 219.79$
Relative Error	$\frac{\hat{C}-C}{C} = -0.025$ %	$\frac{\hat{\mu}-\mu}{\mu} = -0.095$ %

Structure-Preserving ABC: Results for Simulated Data

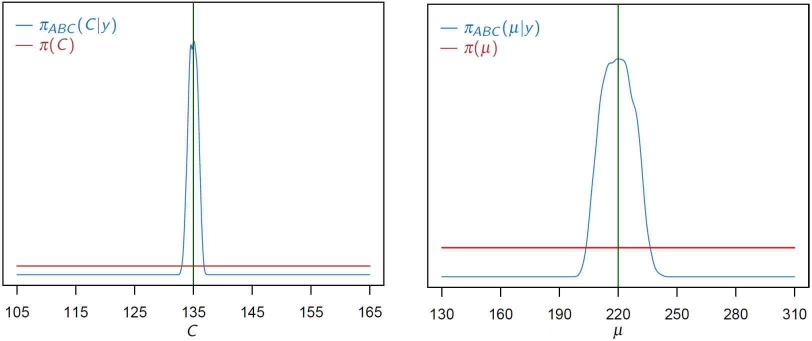

Figure: Marginal ABC posterior densities

	C	μ
Credible Probability CP	$CP = \int_{CI} \pi_{ABC}(C|y)\ dC = 0.997$	$CP = \int_{CI} \pi_{ABC}(\mu|y)\ d\mu = 0.95$
Credible Interval CI	$CI = [133, 137]$	$CI = [205, 235]$

ABC for the JR-NMM: Real EEG α-rhythmic Data[4]

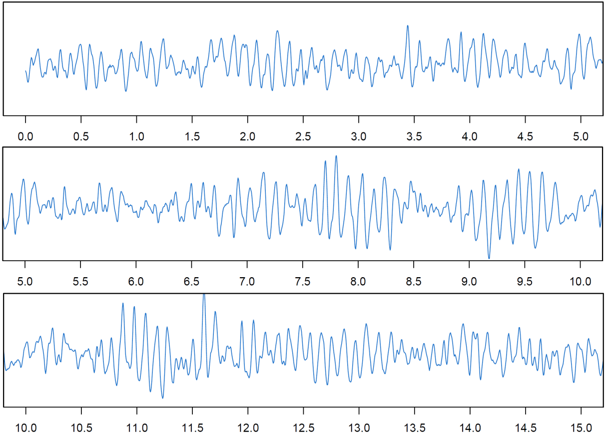

Figure: α-rhythmic EEG segment: sampling rate 173.61 hz

[4]R. Andrzejak, K. Lehnertz, F. Mormann, C. Rieke, P. David, C. Elger. "Indications of nonlinear deterministic and finite-dimensional structures in time series of brain electrical activity: Dependence on recording region and brain state." Physical Review E, 64(6), 8 (2001)

Structure-Preserving ABC: Results for EEG Data

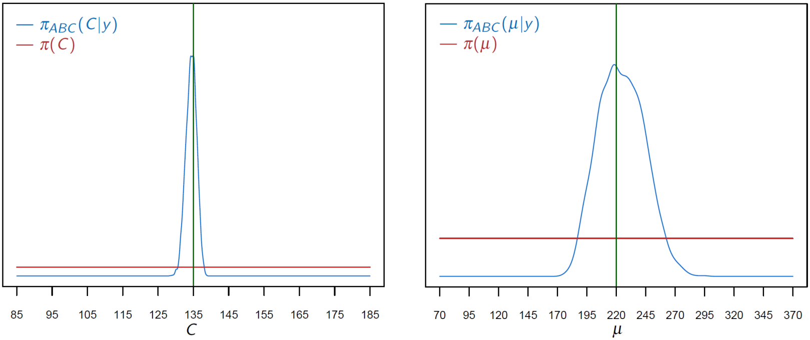

Figure: Marginal ABC posterior densities

- Prior distributions: $\pi(C) = U(85, 185)$, $\pi(\mu) = U(70, 370)$
- Expected values: $C = 135$, $\mu = 220$
- Number of simulated synthetic datasets: $N = 10^6$
- Data generation: time step $\Delta t = 2 \cdot 10^{-3}$, time horizon $T = 23.6$
- Tolerance: $\epsilon = 0.5^{th}$ percentile
- Observed EEG segments of same type: $M = 5$

Structure-Preserving ABC: Results for EEG Data

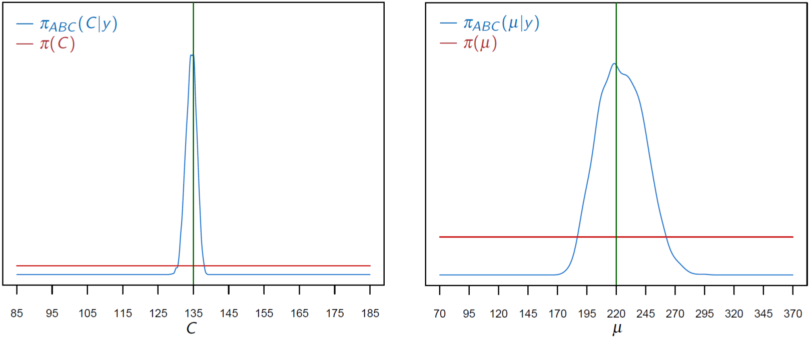

Figure: Marginal ABC posterior densities

	C	μ
ABC Posterior Mean	$\int_D C \cdot \pi_{ABC}(C|y)\, dC = 134.383 = \hat{C}$	$\int_D \mu \cdot \pi_{ABC}(\mu|y)\, d\mu = 224.57$

$$\hat{\theta} = (\hat{C}, \hat{\mu}) = (134.383, 224.578)$$

Structure-Preserving ABC: Results for EEG Data

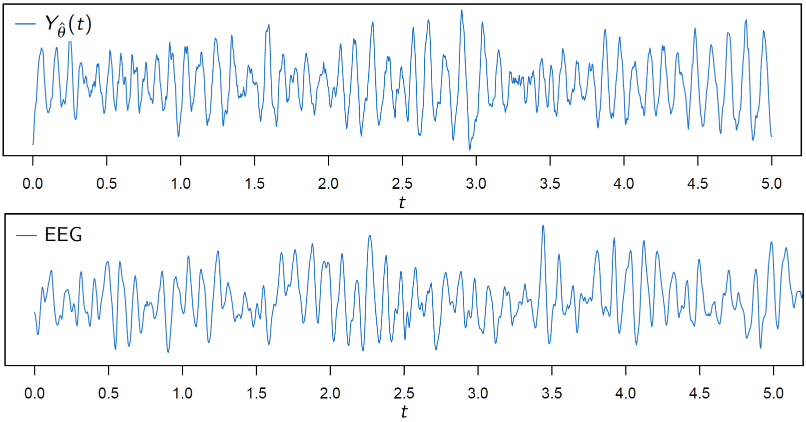

Figure: Sample path for $\hat{\theta} = (\hat{C}, \hat{\mu})$ versus an EEG segment

Non-Preservative ABC: The Method of Euler-Maruyama

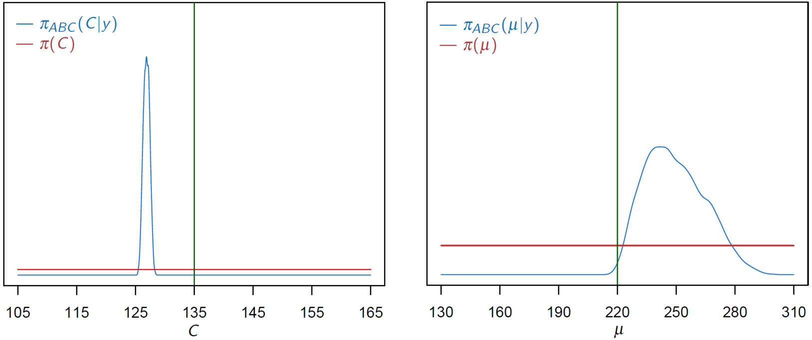

Figure: Marginal ABC posterior densities (simulated reference data)

$\Longrightarrow$ Need for structure-preserving data generation!

Thank you for your attention