Adaptive timestepping for Stochastic (P) DEs

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Stochastic Differential Equations (SDEs) joint with
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&
Stochastic **Partial** Differential Equations joint with :
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Plan :

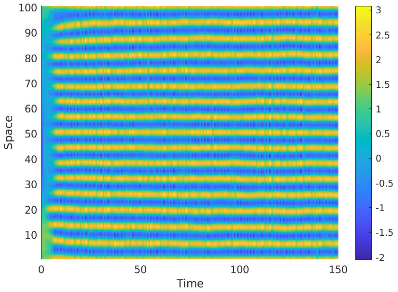
1. Motivation : SPDE & SDE
2. SDE & uniform step methods
3. Introduce Stochastic PDE and uniform step methods
4. Adaptive method & selection of time step
    - Backstop (SPDE example with Multiplicative noise)
      Numerical results
    - A.S. finite N (SPDE example with Additive noise)
      Numerical results
5. Deterministic application ?

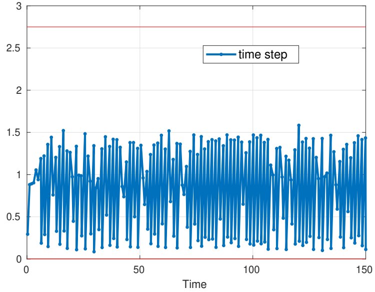
- Deterministic adaptive time stepping : local error control.
- Setting here : adapt for stability.

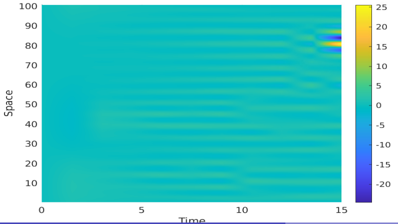
Let's look at some adaptive results

1. Stochastic Swift-Hohenberg - additive noise

$$dX = \beta X - (1+\Delta)^2 X + cX^2 - X^3 dt + BdW$$



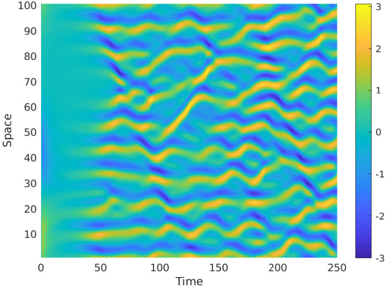


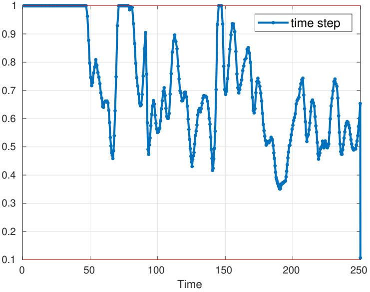


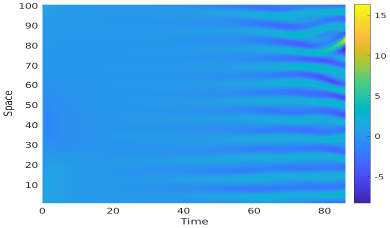
Fixed step $\Delta t = 1.5$.

2. Stochastic Kuramoto-Sivashinsky - multiplicative

$$dX = (-X_{xxxx} - X_{xx} - XX_x)dt + \frac{X}{2}dW$$







Fixed step $\Delta t = 1$.

Motivating SDE Example:

Deterministic ODE with non-globally Lipschitz nonlinearity:

$$X'(t) = -X^3, \quad \text{given } X(0) = X_0, \quad t \geq 0.$$

$X(t) \equiv 0$ is globally asymptotically stable.
Explicit Euler discretization:

$$Y_{n+1} = Y_n - \Delta t Y_n^3, \quad n \in \mathbb{N}.$$

- $Y_n \equiv 0$ locally asy, stable for $Y_0 \in \left(-\sqrt{2/\Delta t}, \sqrt{2/\Delta t}\right)$
- Unstable 2-cycle : $\left\{-\sqrt{2/\Delta t}, \sqrt{2/\Delta t}\right\}$
- If $Y_0 \notin \left[-\sqrt{2/\Delta t}, \sqrt{2/\Delta t}\right]$ then $\lim_{n\to\infty} |Y_n| = \infty$.
- For each fixed $\Delta t > 0$ dynamics is different
- As $\Delta t \to 0$ the scheme converges.

Now include a stochastic perturbation $\cdots$

Motivating Example: Stochastic

Consider the map

$$Y_{n+1} = Y_n - \Delta t Y_n^3 + \underbrace{\Delta\beta_{n+1}}_{:=N(0,\Delta t)}, \quad n \in \mathbb{N}.$$

- For fixed Δt the stochastic perturbation $\Delta\beta_{n+1}$ can push trajectories out of basin of attraction $\left(-\sqrt{2/\Delta t}, \sqrt{2/\Delta t}\right)$
- Problem with growth of Y_n with n!

In this talk we think about changing Δt to Δt_{n+1}.
Idea : Pick a Δt_{n+1} depending Y_n to stay in $\left(-\sqrt{2/\Delta t_n}, \sqrt{2/\Delta t_n}\right)$

In fact β from Browmian motion: $\Delta\beta_{n+1} = (\beta(t_{n+1}) - \beta(t_n))$
Stochastic map is the explicit Euler-Maruyama approximation of SDE

$$X(t_{n+1}) = X(t_n) - \int_{t_n}^{t_{n+1}} X(s)^3 ds + \int_{t_n}^{t_{n+1}} d\beta(s)$$

$$dX(t) = -X(t)^3 + d\beta(t).$$

Euler-Maruyama and growth : (e.g. $f(X) = -X^3$, $g = 1$.)

$$\text{SDE:} \quad dX = f(X)dt + g(X)d\beta.$$

- Suppose f or g
  1. are not globally Lipschitz
  2. and satisfy polynomial growth condition

Then $\mathbb{E}\left[\|X\|^P\right] < \infty$.

$$\text{Euler-Maruyama method:} \quad Y_{n+1} = Y_n + \Delta t f(Y_n) + g(Y_n)\Delta\beta_{n+1}.$$

For numerics would like :
Bounded moments : $\mathbb{E}[\|Y_n\|^p] < \infty$, $p > 0$
Strong convergence : $\mathbb{E}\left[|X(t_n) - Y_n|^2\right] < C\Delta t^q$, $q > 0$.

However
- Fixed step Δt : [Mattingly, Stuart, Higham 2002]
  - Second moment instability :

$$\lim_{n\to\infty} \mathbb{E}\left[|Y_n|^2\right] = \infty.$$

- Non-convergence: [Hutzenthaler, Jentzen, Kloeden 2011].

Some Explicit Methods for SDEs that work ...

- Tamed Methods : Eg [Hutzenthaler et al 2012], [Wang&Gan 2013], [Hutzenthaler&Jentzen 2014], [Sabanis 2013, ...],...
Eg : Drift-tamed Euler-Maruyama

$$Y_{n+1} = Y_n + \frac{\Delta t}{1 + \Delta t \|f(Y_n)\|} f(Y_n) + g(Y_n)\Delta\beta_{n+1}$$

- Basic Idea : Introduce a perturbation

- Balanced Methods : Eg [Tretyakov, Zhang 2013],...
- Truncated Methods : Eg [Mao 2016, Liu& Mao 2017]
- Projected Methods : Eg [Beyn, Isaak, Kruse 2015]

1. Prove Moment bounds

$$\sup_{n \in \mathbb{N}} \sup_{n \in \{0,1,\dots,N\}} \mathbb{E}[\|Y_n\|^p] < \infty.$$

2. Prove strong convergence

$$\left(\mathbb{E}\left[\|X(t) - \bar{Y}_t\|^p\right]\right)^{1/p} \leq C_p \Delta t^{1/2}.$$

Alternatively try adapting the step size.

Stochastic PDE :

We saw at start Stochastic Swift-Hohenberg :

$$dX = \beta X - (1+\Delta)^2 X + cX^2 - X^3 dt + BdW$$

Write our SPDEs as ODE on Hilbert space H :

$$dX = -AX + F(X)dt + B(X)dW$$

We assume :

- $-A : \mathcal{D}(-A) \to H$ the generator of analytic semigroup $S(t) = e^{-tA}, t \geq 0$.
- $B(X)$ globally Lipschitz

$$\|B(X) - B(Y)\|_{L_0^2} \leq L\|X - Y\|, \quad X, Y \in H$$

$$\left\|(-A)^{r/2} B(X)\right\|_{L_0^2} \leq L(1 + \|X\|_r).$$

Stochastic PDE : $dX = -AX + F(X)dt + B(X)dW$

- Define the Wiener process with covariance Q by

$$W(x,t) = \sum_{k=1}^{\infty} \mu_k^{1/2} \phi_k(x) \beta_k(t).$$

- $\beta_k(t)$, be independent identically distributed Brownian motions.
- ϕ_k e.func. of Q, an orthonormal basis of L^2.

(Often assume same e.func. as linear operator $-A$).

- $\mu_k > 0$ are e.values of covariance operator Q for Wiener process.

Determine spatial correlation :
Below :- parameter r. ($r = -0.5$, $Q = I$, $d = 1$).

Note - most applications do not have globally Lipschitz reaction terms F

SPDEs: $dX = -AX + F(X)dt + B(X)dW$

- Mild solution

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)B(X(s))dW(s).$$

  With $S(t) := e^{-tA}$.
- Discretize in space -
  e.g by Finite Elements or spectral Galerkin: $X(t) \approx Y(t)$, $A_h \approx A$.
- Approximation in time to the mild solution:

$$Y(t_{n+1}) = S_h(\Delta t_{n+1})Y(t_n) + \int_{t_n}^{t_{n+1}} S_h(t_{n+1}-s)F(Y(s))ds + \int_{t_n}^{t_{n+1}} S(t_{n+1}-s)B(Y)dW.$$

where, $\Delta t_{n+1} := t_{n+1} - t_n$ and $S_h(\Delta t_{n+1}) := e^{-\Delta t_{n+1} A_h}$.

$$Y_{n+1} := S_h(\Delta t_{n+1})\left(Y_n + \Delta t_{n+1} F(Y_n) + B(Y_n)\Delta W_{n+1}\right)$$

Exponential integrator... still issue with nonlinearity.
(Will also consider semi-implicit).

- Uniform Δt : Many authors : see for example [L & Rougemont], [Jentzen], [Wang], [Cohen], [Tambue], ...

SPDES : Tamed/Stopped methods

With non-globally Lipschitz F, there are four basic approaches :

1. Explicit tamed Euler-Maruyama [Gyongy etal 2016].
   Similar in approach to tamed methods for SDEs.
   Perturbation of F to control growth,
   $$\tilde{F}(X) \approx \frac{F(X)}{1 + \sqrt{\Delta t}\,\|F(X)\|} \tag{1}$$
2. "nonlinearity stopped" method of [Jentzen & Pusnik 2015].
   Exponential integrator with use of indicator function to turn off non-linearities if
   $$\|F(X)\| \geq \left(\frac{1}{\Delta t}\right)^{\theta}, \quad \theta \in (0, \frac{1}{4}]. \tag{2}$$
3. Splitting based methods - often require exact nonlinear flow.
   [Bréhier, Cui & Hong 2019, Bréhier & Goudènege 2019, Cai, Gan & Wang 2021]
4. Adapt the time step ...
   [Campbell & L.], [Hausenblas et al, 2020], [Chen, Dang, Hong]

Adaptive time-stepping:

- Issues from Adaptivity:

1. Increments $\Delta\beta_{n+1}$ depend on Y_n.
   Using that Δt_{n+1} is a bounded $\mathcal{F}_{t_n}$ stopping time by Doob optional sampling theorem [Shirayev 96]

$$\mathbb{E}\left[\Delta\beta_{n+1}|\mathcal{F}_{t_n}\right] = 0 \quad a.s.$$

$$\mathbb{E}\left[|\Delta\beta_{n+1}|^2|\mathcal{F}_{t_n}\right] = \Delta t_{n+1} \quad a.s.$$

2. Random time steps with $t_n = \sum_{j=0}^{n-1} \Delta t_{n+1}$.

   - need to assume each Δt_{n+1} is $\mathcal{F}_{t_n}$ measurable.
   - there is a random integer N to arrive at a final time T.

Adaptive Time-stepping: Upper and Lower bounds

Have random N, Δt_{n+1}

How to ensure we reach our final time T ?

- want finite number of random steps N a.s. and $\Delta t_{n+1} \neq 0$
- need control on Δt_{n+1} to examine convergence.

Hence require that :

$$0 < \Delta t_{n+1} \leq \Delta t_{\max}.$$

Two Approaches : to get to final time T

1. Introduce $\Delta t_{\min}$ and fix deterministic $\rho = \Delta t_{\max}/\Delta t_{\min}$.

$$0 < \Delta t_{\min} \leq \Delta t_{n+1} \leq \Delta t_{\max}.$$

   - When $\Delta t_{n+1} > \Delta t_{\min}$ use the standard method.
   - When $\Delta t_{n+1} \leq \Delta t_{\min}$ Introduce a 'backstop' method and set $\Delta t_{n+1} = \Delta t_{\min}$.

   Example strategy :$\Delta t_{n+1} \leq \Delta t_{\max} \frac{\|Y_n\|}{\|F(Y_n)\|}$

   For SDEs : [Kelly & L, 2017,2018]
   For SPDEs : [Campbell & L.]
   - Can then show $\mathbb{P}\left[\Delta t_{n+1} \leq \Delta t_{\min}\right] < \epsilon$. (See [Kelly, L. & Sun]).

2. For particular strategy for picking Δt_{n+1} show N a.s. finite.
   Example strategy:

$$\Delta t_{n+1} \leq \Delta t_{\max} \frac{(1 + \|Y_n\|^2)}{(1 + \|F(Y_n)\|^2)}.$$

   For SDEs : [Fang & Giles 2016, 2020]
   For McKean Vlasov : [Reisinger & Stockinger, 2021]
   For SPDEs : [Chen, Dang, Hong], [Campbell & L.]

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   For SDEs : [Kelly & L, 2017,2018]
   For SPDEs : [Campbell & L.] (multiplicative noise)

   - Can then show $\mathbb{P}[\Delta t_{n+1} \leq \Delta t_{\min}] < \epsilon$. (See [Kelly, L. & Sun]).

2. For particular strategy for picking Δt_{n+1} show N a.s. finite.
   Example strategy:

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   For SDEs : [Fang & Giles 2016, 2020]
   For McKean Vlasov : [Reisinger & Stockinger, 2021]
   For SPDEs : [Chen, Dang, Hong], [Campbell & L.] (SPDE additive noise)

Backstop Approach : multiplicative noise

$$dX = [-AX + F(X)]dt + B(X)dW$$

On a Hilbert space H with norm $\|.\|$

- Assumptions on F.
  - F satisfies one sided Lipschitz growth condition, $X, Y \in H$

$$\langle F(X) - F(Y), X - Y\rangle \leq L_F \|X - Y\|^2 .$$

$$\|DF(X)\|_{\mathcal{L}(H)} \leq c_1(1 + \|X\|^{c_2}).$$

    for some $L_F, c_1, c_2 > 0$.
- Method :
  - Discretize in space :
    eg spectral Galerkin $Y(t) = \sum_j^J y_j(t)\phi_j(x) \approx X(t)$
  - In time : $Y^n \approx Y(t_n)$
    - $\Delta t_{n+1} > \Delta t_{\min}$: exponential approximation in time.
    - $\Delta t_{n+1} \leq \Delta t_{\min}$: backstop with $\Delta t_{n+1} = \Delta t_{\min}$
      e.g. nonlinear stopped method [Jentzen & Pusnik 2015].

Backstop: $\rho = \Delta t_{\max}/\Delta t_{\min}$.

Example Adaptive Strategy: Pick Δt_{n+1} so that

$$\Delta t_{n+1} \leq \Delta t_{\max} \frac{\|Y_n\|}{\|F(Y_n)\|}.$$

- $\Delta t_{n+1} < \Delta t_{\min}$ then we use a backstop method
- $\Delta t_{n+1} \geq \Delta t_{\min}$ then use standard exponential method.

$$\|F(Y_n)\| \leq \frac{\Delta t_{\max}}{\Delta t_{n+1}} \|Y_n\| \leq \rho \|Y_n\|.$$

To bound non-global Lipschitz nonlinearity: (avoid bound on $\mathbb{E}\left[\|Y_n\|^p\right]$).

$$\begin{aligned} \|F(Y_n) - F(X(t_n))\|^2 &\leq 2\|F(Y_n)\|^2 + 2\|F(X(t_n))\|^2 \\ &\leq 2\rho^2 \|Y_n\|^2 + 2\|F(X(t_n))\|^2 \end{aligned}$$

Now add in and subtract $X(t_n)$ so that $Y_n = X(t_n) - Y_n - X(t_n)$

$$\|F(Y_n) - F(X(t_n))\|^2 \leq 4\rho^2 \|E_n\|^2 + 4\rho \|X(t_n)\|^2 + 2\|F(X(t_n))\|^2$$

Strong Convergence [Stuart Campbell, L.]

Let $X(T)$ be the mild solution to SPDE.
Let Y_N be the numerical approximation defined over $\{t_n\}_{n\in\mathbb{N}}$, an admissible time-stepping strategy.
For $X_0 \in L^2(\mathbb{D}, \mathcal{D}((-A)^{1/2}))$, $\epsilon > 0$

- Multiplicative noise : $r \in (0,1)$

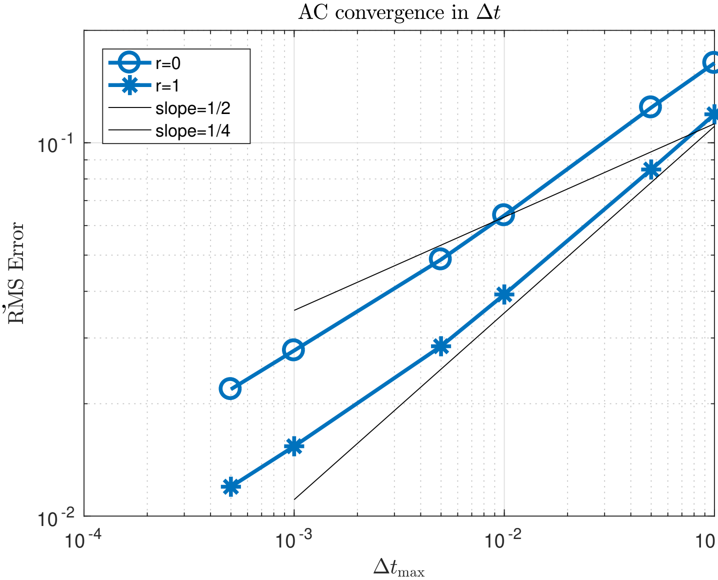
$$\left(\mathbb{E}\left\|X(T) - Y_N^h\right\|^2\right)^{1/2} \leq C(T)(\Delta x^{1+r} + \Delta t_{\max}^{\frac{1}{2}-\epsilon} + \lambda_{M+1}^{-\frac{1+r}{2}+\epsilon}).$$

(restrictive conditions on nonlinearity - eg not $X - X^3$).
Proof : outline

- Need to deal with conditional expectation.
  E.g. to use $\mathbb{E}\left[|\Delta\beta_{n+1}|^2|\mathcal{F}_{t_n}\right] = \Delta t_{n+1} \quad a.s.$
- Need to look at error over 1-step (not final time estimate)
- Need to combine adaptive scheme and backstop and deal with random number of steps N.

$$dX = \Delta X + X - X^3 dt + BXdW$$

$$x = [0, 2\pi],$$
$$\rho = 100,$$
$$T = 5,$$
$$\Delta t_n \leq \frac{\Delta t_{\max}}{\|F(X)\|},$$
$$N_x = 512,$$
$$B = 1,$$
$$r = 0.$$



Numerical Methods

Compare 4 numerical methods

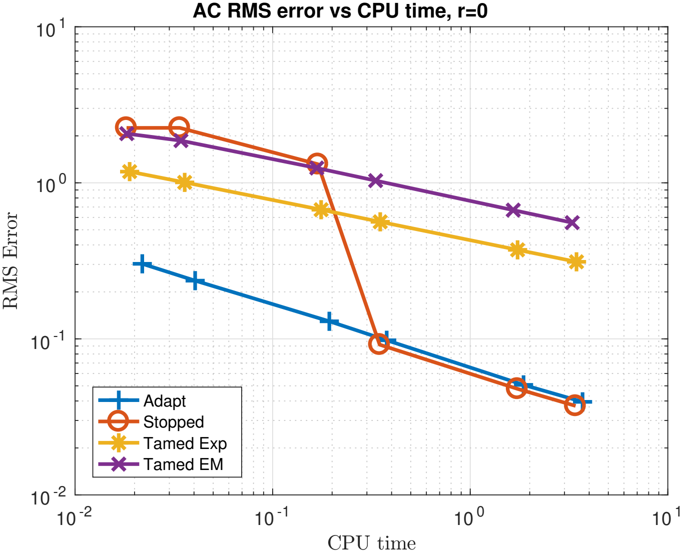
- Adaptive
  $Y^h_{n+1} = S_h(\Delta t_{n+1})\left(Y^h_n + F(Y^h_n)\Delta t_{n+1} + B(Y^h_n)\Delta W_{n+1}\right)$
- Stopped
  $Y^h_{n+1} =$
  $S_h(\Delta t)\left(Y^h_n + \left\{F(Y^h_n)\Delta t + B(Y^h_n)\Delta W_{n+1}\right\}\mathbb{1}_{\left\|F(Y^h_n)\right\|\le\left(\frac{1}{\Delta t}\right)^\theta}\right)$
- Tamed Exponential (no proof)
  $Y^h_{n+1} = S_h(\Delta t)\left(Y^h_n + \tilde{F}(Y^h_n)\Delta t + B(Y^h_n)\Delta W_{n+1}\right)$
- Tamed Euler-Maruyama
  $Y^h_{n+1} = Y^h_n + \tilde{C}(Y^h_n)\Delta t + B(Y^h_n)\Delta W_{n+1}$

where $C(X) = -AX + F(X)$ and $\tilde{f}(X) = \frac{f(X)}{1+\sqrt{\Delta t}\|f(X)\|}$.

For fixed step methods set $\Delta t = \overline{\Delta t} = \frac{1}{N}\sum \Delta t_n$

$dX = \Delta X + X - X^3 dt + BXdW$

$$x = [0, 2\pi],$$
$$\rho = 100,$$
$$T = 5,$$
$$\Delta t_n \leq \frac{\Delta t_{\max}}{\|F(X)\|},$$
$$N_x = 512,$$
$$B = 1,$$
$$M = 100.$$



SPDE - Additive noise

$$dX = [-AX + F(X)]dt + BdW$$

On a Hilbert space H with norm $\|.\|$.
Assumption on F

- F satisfies one sided Lipschitz growth condition, $X, Y \in H$

$$\langle F(X) - F(Y), X - Y \rangle \leq L_F \|X - Y\|^2 .$$

$$\|F(X) - F(Y)\| \leq C(1 + \|X\|_E^c + \|Y\|_E^c)\|X - Y\|.$$

$$\|DF(X)\|_{\mathcal{L}(H)} \leq C(1 + \|X\|_E^c)\|$$

$$\|F(X)\|_E \leq C(1 + \|X\|_E^c), \quad \|F(X)\| \leq C(1 + \|X\|_E^c)\|X\|,$$

where $\|u\|_E := \sup_{x \in D} |u(x)|$.

Here can look at, for example, Allen-Cahn equation $F(X) = X - X^3$.

Showing N a.s. finite

$$dX = [-AX + F(X)]dt + BdW$$

- Discretize in space :
  eg spectral Galerkin $Y(t) = \sum_j y_j(t)\phi_j(x) \approx X(t)$
- In time : $Y(t_n) \approx Y_n$ from exponential method.
  We have $T = \sum_{j=0}^{N} \Delta t_{n+1}$. Need N a.s. finite.

$$0 < \Delta t_{n+1} \leq \Delta t_{\max} \frac{(1 + \|Y_h^n\|^2)}{(1 + \|F(Y_h^n)\|^2)}.$$

Our starting point : we know we can do K steps. Prove that must reach T

Other see : [Fang & Giles 2020] for SDEs and [Chen, Dang, Hong] for SPDEs.

Showing N a.s. finite

Adaptive exponential method is defined by the recursion

$$Y^{n+1} = \underbrace{S_h(\Delta t_{n+1}) P_h Y^n + \int_{t_n}^{t_{n+1}} S_h(t_{n+1} - s) P_h F(Y^n) ds}_{Z^n}$$

$$+ \underbrace{\int_{t_n}^{t_{n+1}} S_h(t_{n+1} - t_n) P_h B P_J dW(s)}_{W^n}.$$

1. Bound $\mathbb{E}[\|W^n\|^p]$ and $\mathbb{E}[\|F(W^n)\|^p]$ for all n
2. Z^n : use adaptivity to bound $\mathbb{E}\left[\|Z^K\|^p\right]$ after K deterministic steps.
3. Use dominated convergence to bound $\mathbb{E}\left[\left\|Z^N\right\|^p\right] = \mathbb{E}\left[\lim_{K\to\infty} \left\|Z^K(\tau_K)\right\|^p\right]$ independently of K, N, $\tau_K := \sum_{n=0}^{N} \Delta t_{n+1} \mathbb{1}_{\{n \le K\}}$.
4. Timestepping plus moment bounds form a contradiction argument so
   - $\exists$ a.s. finite N
   - with $\mathbb{E}[\tau_N] = T$,
   - and $\mathbb{E}[N] = O(1/\Delta t_{\max})$.
5. Finite upper bound on T and reverse Markov shows $\mathbb{P}[\tau_N < T] = 0$.

Strong Convergence [Stuart Campbell, L.]

Let $X(T)$ be the mild solution to SPDE.
Let Y_N^h be the numerical approximation defined over $\{t_n\}_{n\in\mathbb{N}}$, an admissible time-stepping strategy.
For $X_0 \in L^2(\mathbb{D}, \mathcal{D}((-A)^{1/2}))$, $\epsilon > 0$

- Additive noise : $r \in (-1, 0]$

$$\left(\mathbb{E}\left\|X(T) - Y_N^h\right\|^2\right)^{1/2} \leq C(T)(\Delta x^{1+r-\epsilon} + \Delta t_{\max}^{\min(\frac{1}{2},(1+r)/2)-\epsilon} + \lambda_{M+1}^{-\frac{1+r}{2}+\epsilon}).$$

Notes:

- less restrictive conditions on nonlinearity: eg $X - X^3$ OK.
- includes space-time white.

Proof : Use that have finite N a.s. and moment bound.

Numerical Methods

Compare 4 numerical methods

- Adaptive
  $Y^h_{n+1} = S_h(\Delta t_{n+1})\left(Y^h_n + F(Y^h_n)\Delta t_{n+1} + B\Delta W_{n+1}\right)$
- Stopped
  $$Y^h_{n+1} = S_h(\Delta t)\left(Y^h_n + \left\{F(Y^h_n)\Delta t + B\Delta W_{n+1}\right\}\mathbb{1}_{\|F(Y^h_n)\| \le \left(\frac{1}{\Delta t}\right)^\theta}\right)$$
- Tamed Exponential (no proof)
  $$Y^h_{n+1} = S_h(\Delta t)\left(Y^h_n + \tilde{F}(Y^h_n)\Delta t + B\Delta W_{n+1}\right)$$
- Tamed Euler-Maruyama
  $Y^h_{n+1} = Y^h_n + \tilde{C}(Y^h_n)\Delta t + B\Delta W_{n+1}$

where $C(X) = -AX + F(X)$ and $\tilde{f}(X) = \frac{f(X)}{1+\sqrt{\Delta t}\|f(X)\|}$.

For fixed step methods set $\Delta t = \overline{\Delta t} = \frac{1}{N}\sum \Delta t_n$

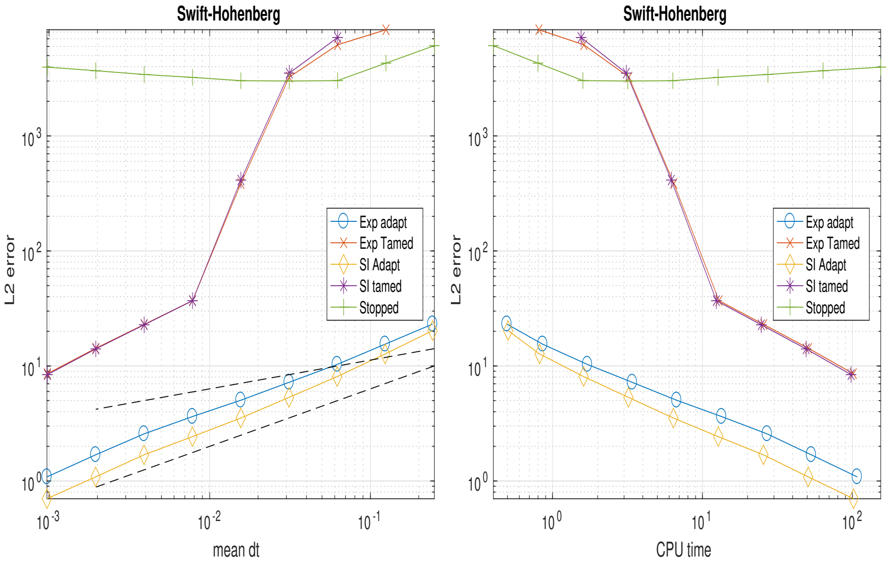
Swift-Hoenberg SPDE

- SPDE defined by

$$dX = (\beta X - (1+\Delta)^2 X + cX^2 - X^3)dt + BdW,$$

  we set $\beta = -0.7$, $c = 1.8$ and $B = 0.5$.
- Used in many applications involving pattern formation, including fluid flow and neural tissue.

$$dX = \beta X - (1+\Delta)^2 X + cX^2 - X^3 dt + BdW \ (r = -0.5)$$



Summary so far

- Introduced issue of non-convergence for explicit methods
  - SDE
  - Stochastic PDEs
- Adaptive time stepping :
  - Conditional Expectation to recover standard Brownian motion properties.
  - Need $0 < \Delta t_{n+1}$ and finite N a.s.
    Two strategies
  - Used Backstop strategy - for multiplicative noise.
    Examined strong convergence
  - Proof of N a.s. Finite - for additive noise.
    Examined strong convergence
- In both cases see improved efficiency

Application in deterministic setting ?

Given

$$dX = -AX + F(X)dt + B(X)dW$$

Examined exponential integrator:

$$Y_{n+1} := S_h(\Delta t_{n+1})\left(Y_n + \Delta t_{n+1}F(Y_n) + B(Y_n)\Delta W_{n+1}\right)$$

where, $\Delta t_{n+1} := t_{n+1} - t_n$ and $S_h(\Delta t_{n+1}) := e^{-\Delta t_{n+1}A_h}$.
Alternative : semi-implicit

$$Y_{n+1} := (I + \Delta t A)^{-1}\left(Y_n + \Delta t_{n+1}F(Y_n) + B(Y_n)\Delta W_{n+1}\right)$$

Similar results on the adaptivity.
In deterministic setting $B \equiv 0$:
Get standard exponential integrator

$$Y_{n+1} := S_h(\Delta t_{n+1})\left(Y_n + \Delta t_{n+1}F(Y_n)\right)$$

Or semi-implicit method

$$Y_{n+1} := (I + \Delta t_{n+1}A)^{-1}\left(Y_n + \Delta t_{n+1}F(Y_n)\right)$$

Deterministic case

Standard exponential integrator

$$Y_{n+1} := S_h(\Delta t_{n+1})\left(Y_n + \Delta t_{n+1} F(Y_n)\right)$$

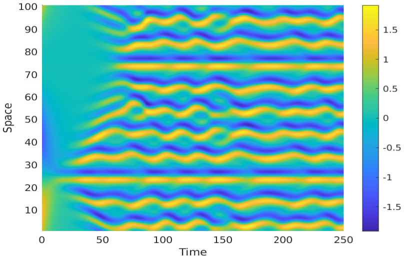
Or semi-implicit method

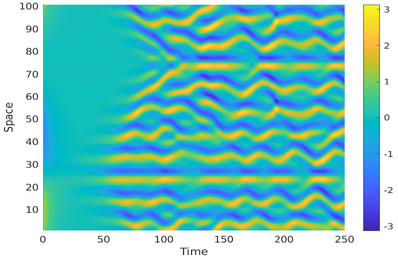
$$Y_{n+1} := (I + \Delta t_{n+1} A)^{-1}\left(Y_n + \Delta t_{n+1} F(Y_n)\right)$$

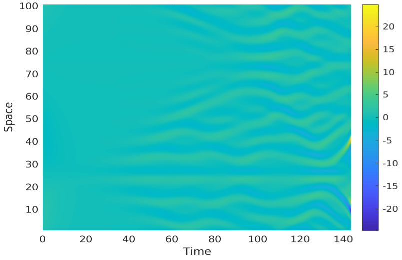
- There is no instability directly from from the linear term.
- But nonlinearity is explicit.
- Have a restriction on Δt from the nonlinearity.

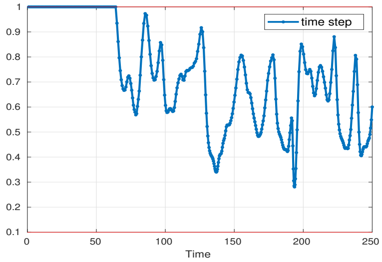
Deterministic KS : $u_t = -u_{xxxx} - u_{xx} - uu_x$

$\Delta t = 0.1, \Delta t = 0.6702$ $\Delta t_{\max} = 1$



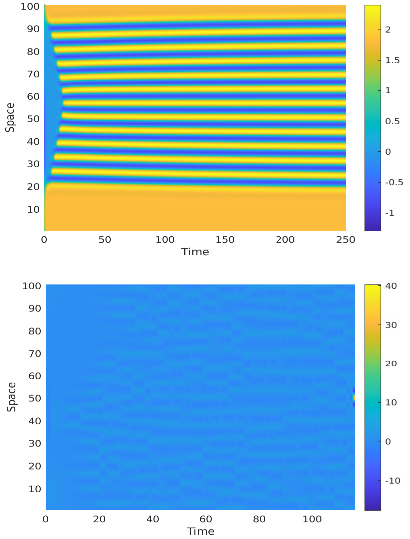




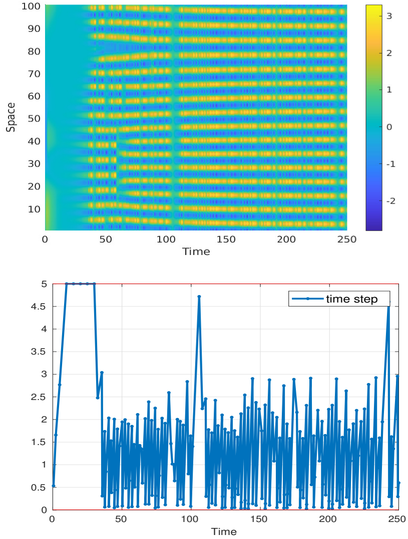


Deterministic SH : $u_t = \beta u - (1+\Delta)^2 u + cu^2 - u^3$

$\Delta t = 0.1, \Delta t = 1.2077$



$\Delta t_{\max} = 5$



Summary ... again

- Introduced issue of non-convergence for explicit methods
  - SDE
  - Stochastic PDEs
- Adaptive time stepping :
  - Conditional Expectation to recover standard Brownian motion properties.
  - Need $0 < \Delta t_{n+1}$ and finite N a.s.
    Two strategies
  - Used Backstop strategy - for multiplicative noise.
    Examined strong convergence
  - Proof of N a.s. Finite - for additive noise.
    Examined strong convergence
  - In both cases see improved efficiency
- Potential application for deterministic system.

- Thank you.