A pressure-robust discretization of the Stokes problem on anisotropic meshes

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Outline

Plan of the talk

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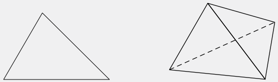
Shape of finite elements

Shape-regular elements



- shape regularity: $h_T \sim \varrho_T$

$$h_T = \text{diam} T, \quad \varrho_T = \sup_{B \subset T} \text{diam} B$$

- minimal angle condition [Zlamal 1968, Ženišek 1968]:

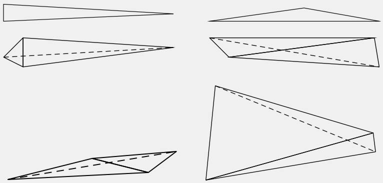
$$\exists\, \alpha_0 > 0 : \alpha_T \geq \alpha_0 \quad \forall T \in \mathcal{T}_h$$

α_T is the minimal angle in T

- majority of papers

Anisotropic elements

- large aspect ratio h_T / ϱ_T



Variants of assumptions:

- regular vertex property
- maximal angle condition
- no assumption on shape

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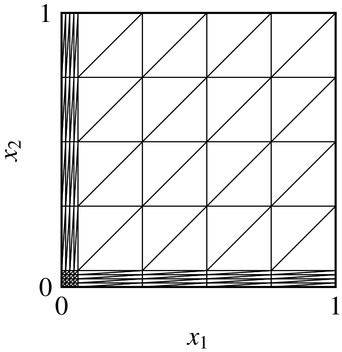
Use cases of anisotropic mesh grading

Boundary layers

- Singularly perturbed problems, e.g., $-\epsilon^2\Delta u + u = f$, $0 < \epsilon \ll 1$: $\delta \sim \epsilon|\ln\epsilon|$
- Flow problems, e.g., $\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}$, $0 < \nu \ll 1$: $\delta \sim \sqrt{\nu}$
- Anisotropy in mesh depends on perturbation parameter ϵ / ν



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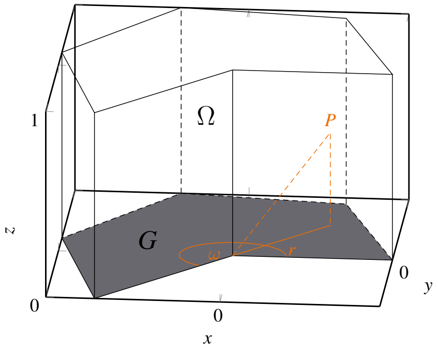
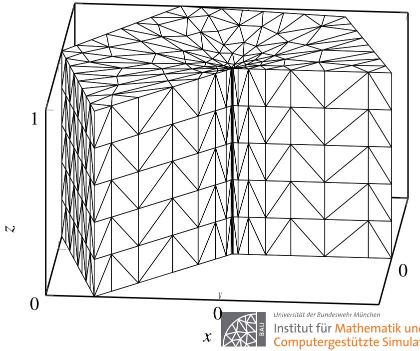
Use cases of anisotropic mesh grading

Edge singularities

- Poisson problem, $-\Delta u = f$: solution has singular part r^λ, $\lambda = \frac{\pi}{\omega}$
- Flow problems, e.g., $-\nu\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}$: solution has singular part r^λ, λ is smallest positive solution of $\sin(\lambda\omega) = -\lambda\sin(\omega)$
- Anisotropy in mesh depends on mesh size parameter h: $h_x \sim h^{1/\mu}$





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Challenge with anisotropic elements

Constants:

- Isotropic elements: The constant in certain estimates may depend on the aspect ratio h_T/ϱ_T.
- Anisotropic elements: Constants must not depend on the aspect ratio.
  This may or may not be possible.
  In the positive case it usually requires a refined proof.

Example for the difficulty: Piola transformation

$$\boldsymbol{x} = J_T\hat{\boldsymbol{x}} + \boldsymbol{x}_0 = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\hat{\boldsymbol{x}} + \boldsymbol{x}_0 \quad \Rightarrow \quad \boldsymbol{v} = \frac{1}{\det J_T} J_T\hat{\boldsymbol{v}} = \begin{pmatrix} h_2^{-1} & 0 \\ 0 & h_1^{-1} \end{pmatrix}\hat{\boldsymbol{v}}$$

Plan of the talk

1. Anisotropic elements
2. Pressure-robustness
3. Interpolation error estimates for RT and BDM elements
4. Discretization error estimate for CR with reconstruction
5. Numerical example

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Stokes equations

Saddle point formulation

Find $(\boldsymbol{u}, p) \in \boldsymbol{X} \times Q = \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}) \quad && \forall \boldsymbol{v} \in \boldsymbol{X}, \\ (\nabla \cdot \boldsymbol{u}, q) &= 0 && \forall q \in Q. \end{aligned}$$

$\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, ν kinematic viscosity, $(\cdot,\cdot)$ L^2-scalar product.

Reduced formulation

Find $\boldsymbol{u} \in \boldsymbol{X}^0 = \{\boldsymbol{v} \in \boldsymbol{X} : (\nabla \cdot \boldsymbol{v}, q) = 0 \ \forall q \in Q\}$ such that

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}^0.$$

Velocity $\boldsymbol{u} \in \boldsymbol{X}^0$ is completely determined by the divergence-free test functions

Pressure-robustness

Helmholtz decomposition

L^2-orthogonal decomposition into a divergence-free and an irrotational part:

$$\boldsymbol{f} = \mathbb{P}\boldsymbol{f} + \nabla\phi$$

divergence-free test functions remove $\nabla\phi$:

$$(\boldsymbol{f}, \boldsymbol{v}) = (\mathbb{P}\boldsymbol{f}, \boldsymbol{v}) + (\nabla\phi, \boldsymbol{v}) = (\mathbb{P}\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}^0$$

Reduced formulation

Find $\boldsymbol{u} \in \boldsymbol{X}^0 = \{\boldsymbol{v} \in \boldsymbol{X} : (\nabla \cdot \boldsymbol{v}, q) = 0 \; \forall q \in Q\}$ such that

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) = (\mathbb{P}\boldsymbol{f}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}^0.$$

only divergence-free part of $\boldsymbol{f}$ determines $\boldsymbol{u}$

Example: $\boldsymbol{f} = \nabla\psi$ leads to $\boldsymbol{u} = \boldsymbol{0}$ and $p = \psi$

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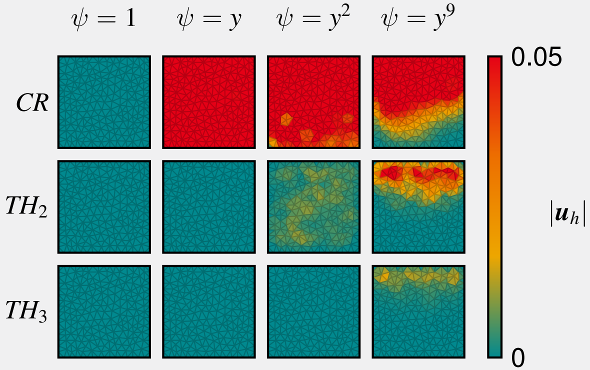
Pressure-robustness

Test problem

Stokes equations with
$\boldsymbol{f} = \nabla\psi$.

$$\Rightarrow \quad \boldsymbol{u} \equiv \boldsymbol{0}, \qquad p = \psi.$$



- $\boldsymbol{u} \equiv \boldsymbol{0}$ is contained in any approximation space $\boldsymbol{X}_h$
- $\boldsymbol{u}_h \equiv 0$ if method is pressure robust
- here: only if $p = \psi \in Q_h$ then $\boldsymbol{u}_h \equiv 0$ (and $p_h = p$)
- otherwise $\boldsymbol{u}_h \not\equiv 0$ since methods are not pressure robust

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Pressure-robustness

Many references:

1. $\boldsymbol{X}^0$-conforming methods
2. $\boldsymbol{H}_0(\mathrm{div}, \Omega)$-conforming methods
3. grad-div stabilization
4. divergence-free reconstruction operator

Pressure-robustness

Crouzeix–Raviart method

$\boldsymbol{X}_h$ p.w. linear, continuous at barycenters of facets, Q_h p.w. constant

- inf-sup stable on arbitrary meshes[1]
- necessary approximation and interpolation properties satisfied

Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times Q_h$ such that

$$\begin{aligned} \nu(\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{v}_h) - (\nabla_h \cdot \boldsymbol{v}_h, p_h) &= (\boldsymbol{f}, \boldsymbol{v}_h) \quad && \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \\ (\nabla_h \cdot \boldsymbol{u}_h, q_h) &= 0 && \forall q_h \in Q_h. \end{aligned}$$

Not pressure-robust

- $(\boldsymbol{f}, \boldsymbol{v}_h) \neq (\mathbb{P}\boldsymbol{f}, \boldsymbol{v}_h)$, since for $\boldsymbol{v}_h \in \boldsymbol{X}_h^0$ in general $\boldsymbol{v}_h \notin \boldsymbol{X}^0$ with $\boldsymbol{X}_h^0 = \{\boldsymbol{v}_h \in \boldsymbol{X}_h : (\nabla_h \cdot \boldsymbol{v}_h, q_h) = 0 \; \forall q_h \in Q_h\}$
- $\boldsymbol{L}^2$-orthogonality $(\nabla\phi, \boldsymbol{v}_h)$ is not satisfied

[1] T. Apel, S. Nicaise, and J. Schöberl. “A non-conforming finite element method with anisotropic mesh grading for the Stokes problem in domains with edges”. *IMA J. Numer. Anal.* 21.4 (2001), 843–856

The reconstruction approach

Recover L^2-orthogonality through reconstruction[2]

Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times Q_h$ such that

$$\begin{aligned} \nu(\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{v}_h) - (\nabla_h \cdot \boldsymbol{v}_h, p_h) &= (\boldsymbol{f}, I_h \boldsymbol{v}_h) \quad && \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \\ (\nabla_h \cdot \boldsymbol{u}_h, q_h) &= 0 && \forall q_h \in Q_h. \end{aligned}$$

- I_h maps test functions $\boldsymbol{v}_h \in \boldsymbol{X}_h^0$ to divergence-free functions
- $\Rightarrow (\boldsymbol{f}, I_h \boldsymbol{v}_h) = (\mathbb{P}\boldsymbol{f}, I_h \boldsymbol{v}_h)$ for all $\boldsymbol{v}_h \in \boldsymbol{X}_h^0$
- $\Rightarrow$ we seek $\boldsymbol{u}_h \in \boldsymbol{X}_h^0$ such that

$$\nu(\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{v}_h) = (\mathbb{P}\boldsymbol{f}, I_h \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h^0$$

which means pressure robustness

$$\boldsymbol{X}_h^0 = \{\boldsymbol{v}_h \in \boldsymbol{X}_h : (\nabla_h \cdot \boldsymbol{v}_h, q_h) = 0 \; \forall q_h \in Q_h\}$$

[2] A. Linke. "On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime". *Comput. Methods Appl. Mech. Engrg.* 268 (2014), 782–800

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Reconstruction operators for Crouzeix–Raviart method

Lowest order RT/BDM interpolation: Properties

- preserve discrete divergence
- necessary interpolation property satisfied[3,4]

$$\|\boldsymbol{u} - I_h\boldsymbol{u}\|_{0,T} \leq ch_T\|D^1\boldsymbol{u}\|_{0,T}$$

with $I_h \in \{\mathrm{RT}_0, \mathrm{BDM}_1\}$

[3] G. Acosta, T. Apel, R. G. Durán, and A. L. Lombardi. "Error estimates for Raviart–Thomas interpolation of any order on anisotropic tetrahedra". *Math. Comp.* 80.273 (2011), 141–163

[4] T. Apel and V. Kempf. "Brezzi–Douglas–Marini interpolation of any order on anisotropic triangles and tetrahedra". *SIAM J. Numer. Anal.* 58.3 (2020), 1696–1718

Plan of the talk

Elements

For the talk: $H(\mathrm{div})$-conforming elements

RT: Raviart-Thomas elements

$$\mathcal{RT}_k(T) = \boldsymbol{P}_k(T) + \boldsymbol{x}\, P_k(T),$$

Interpolation:

$$\int_{f_i} (I_k \boldsymbol{v}) \cdot \boldsymbol{n}_i z = \int_{f_i} \boldsymbol{v} \cdot \boldsymbol{n}_i z, \quad \forall z \in P_k(f_i)$$

$$\int_T (I_k \boldsymbol{v}) \cdot \boldsymbol{z} = \int_T \boldsymbol{v} \cdot \boldsymbol{z}, \qquad \forall \boldsymbol{z} \in \boldsymbol{P}_{k-1}(T)$$

$k \geq 0$, [Rav./Th. 77], [Nedelec 80]

BDM: Brezzi-Douglas-Marini

$$\mathcal{BDM}_k(T) = \boldsymbol{P}_k(T)$$

Interpolation:

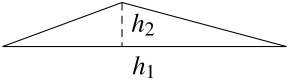
$$\int_{f_i} (I_k \boldsymbol{v}) \cdot \boldsymbol{n}_i z = \int_{f_i} \boldsymbol{v} \cdot \boldsymbol{n}_i z, \quad \forall z \in P_k(f_i)$$

$$\int_T (I_k \boldsymbol{v}) \cdot \boldsymbol{z} = \int_T \boldsymbol{v} \cdot \boldsymbol{z}, \qquad \forall \boldsymbol{z} \in \boldsymbol{N}_{k-1}(T)$$

$k \geq 1$, [Nedelec 86]

original definition [BDM 85] differs, less suitable for anisotropic elements

Interpolation error estimates, no shape assumption



- multiindex notation $\alpha = (\alpha_1, \dots, \alpha_d)$

$$h^\alpha = \prod_{i=1}^{d} h_i^{\alpha_i}, \quad D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$$

- error estimate $(d = 2)$

$$\|\boldsymbol{v} - I_1 \boldsymbol{v}\|_{0,T} \lesssim \sum_{|\alpha| \le 1} h^\alpha \|D^\alpha v_1\|_{0,T} \left(1 + \frac{h_2}{h_1}\right) + \sum_{|\alpha| \le 1} h^\alpha \|D^\alpha v_2\|_{0,T} \left(1 + \frac{h_1}{h_2}\right)$$

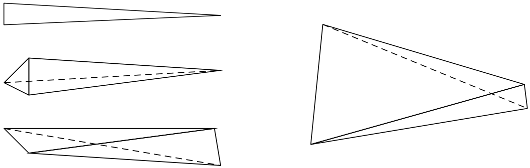
- right hand side depends on the aspect ratio
- sharper estimates under shape assumptions

Maximal angle condition

- Maximal angle condition:

$$\exists\,\gamma_0 < \pi : \gamma_T \leq \gamma_0 \quad \forall T \in \mathcal{T}_h$$

  where γ_T is the maximal angle in T (between and within facets)
- positive examples:



- negative examples:

- 2D: [Synge 1957], [Babuška/Aziz 1976], [Barnhill/Gregory 1976], [Jamet 1976]
- 3D: [Křížek 1992], equivalent definitions: [Jamet 1976], [Apel/Dobrowolski 1992]

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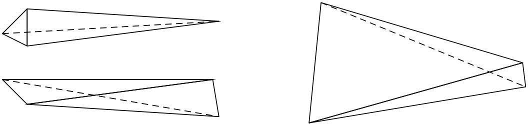
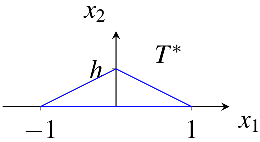
Interpolation error estimates, maximal angle condition



- with refined proof:

$$\|\boldsymbol{v} - I_k\boldsymbol{v}\|_{0,T} \lesssim h_T^{k+1} \left\| D^{k+1}\boldsymbol{v} \right\|_{0,T}$$

  - RT: [Acosta/Duran 99], [Duran/Lombardi 08], [Acosta/Apel/Duran/Lombardi 11]
  - BDM: [Apel/Kempf 21]
- necessity of the maximal angle condition can be shown by an example



$$\boldsymbol{v}(\boldsymbol{x}) = (0, x_1^2)^T \quad \Rightarrow \quad \|I_1\boldsymbol{v}\|_{0,T^*} = O(h^{-1/2})$$

$$\|v_2\|_{0,T^*} \sim |v_2|_{1,T^*} = O(h^{1/2})$$

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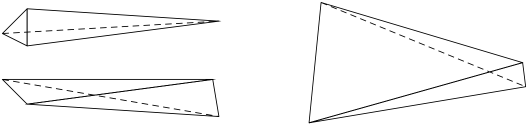
Interpolation error estimates, maximal angle condition



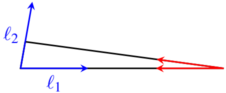
- with refined proof:

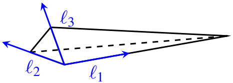
$$\|\boldsymbol{v} - I_k\boldsymbol{v}\|_{0,T} \lesssim h_T^{k+1} \left\| D^{k+1}\boldsymbol{v} \right\|_{0,T}$$

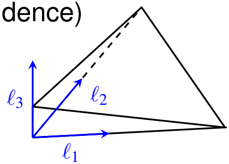
  - RT: [Acosta/Duran 99], [Duran/Lombardi 08], [Acosta/Apel/Duran/Lombardi 11]
  - BDM: [Apel/Kempf 21]
- necessity of the maximal angle condition can be shown by an example
- only the diameter enters the estimate
- sharper estimates under the regular vertex property

Regular vertex property (RVP)

- There is one vertex where the directions of the adjacent edges form a stable coordinate system (uniform linear independence) [Acosta/Duran 2000]



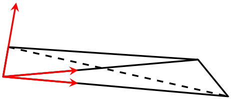




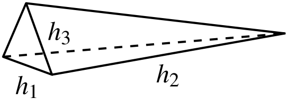
- 2D: equivalent to maximal angle condition
- 3D: one can require the RVP in the case of flat tetrahedra

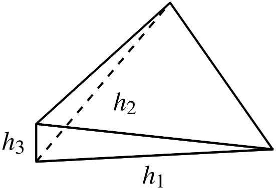
  but in the case of needle elements one cannot fill the space with such tetrahedra exclusively;

  one needs tetrahedra without regular vertex



Interpolation error estimates, regular vertex property





- with refined proof:

$$\|\boldsymbol{v} - I_k \boldsymbol{v}\|_{0,T} \lesssim \sum_{|\alpha|=k+1} h^\alpha \|D_\ell^\alpha \boldsymbol{v}\|_{0,T} + h_T^{k+1} \left\| D^k \operatorname{div} \boldsymbol{v} \right\|_{0,T}$$

  - RT: [Acosta/Duran 99], [Duran/Lombardi 08], [Acosta/Apel/Duran/Lombardi 11]
  - BDM: [Apel/Kempf 21]
- necessity of the regular vertex property can be shown by an example
- remedy for needle elements: triangular prisms
- [Farhloul/Nicaise/Paquet 01]: RT, $k = 0$: first interpolate to the RT space on anisotropic prisms and then interpolate to the simplicial partition of these prisms – does not work for BDM

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Hierarchy of shape assumptions

Possible assumptions on triangles and tetrahedra are:

- not any
- maximal angle condition
- regular vertex property
- minimal angle condition = shape regularity

Anisotropic interpolation error estimates:

- for conforming Lagrangian elements with maximal angle condition
- for $H(\mathrm{div})$-conforming elements with regular vertex property

Plan of the talk

1. Anisotropic elements
2. Pressure-robustness
3. Interpolation error estimates for RT and BDM elements
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Stokes equations

Weak form

Find $(\boldsymbol{u}, p) \in \boldsymbol{X} \times Q = \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{aligned} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) &= (\boldsymbol{f}, \boldsymbol{v}) \quad && \forall \boldsymbol{v} \in \boldsymbol{X}, \\ (\nabla \cdot \boldsymbol{u}, q) &= 0 && \forall q \in Q. \end{aligned}$$

$\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, ν kinematic viscosity, $(\cdot, \cdot)$ L^2-scalar product.

Discrete weak formulation with CR and reconstruction

Find $\boldsymbol{u}_h \in \boldsymbol{X}_h^0 = \{\boldsymbol{v}_h \in \boldsymbol{X}_h : (\nabla_h \cdot \boldsymbol{v}_h, q_h) = 0 \; \forall q_h \in Q_h\}$ such that

$$\nu(\nabla_h \boldsymbol{u}_h, \nabla_h \boldsymbol{v}_h) = (\boldsymbol{f}, I_h \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h^0.$$

The reconstruction operator I_h is the RT or BDM interpolant.

Error estimates on anisotropic mesh, full regularity

Pressure-robust estimates[5]

Let Ω be convex such that $(\boldsymbol{u}, p) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$, and let $\mathcal{T}_h$ satisfy a maximum angle condition. Then the pressure-robust estimates

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} \leq ch|\boldsymbol{u}|_2 \qquad \text{for both reconstructions}$$
$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \leq ch^2|\boldsymbol{u}|_2 \qquad \text{for BDM reconstruction}$$

hold.

For the pressure, we get

$$\|p - p_h\|_0 \leq ch(|p|_1 + |\boldsymbol{u}|_2).$$

[5] T. Apel, V. Kempf, A. Linke, and C. Merdon. "A nonconforming pressure-robust finite element method for the Stokes equations on anisotropic meshes". *IMA J. Numer. Anal.* (2021)

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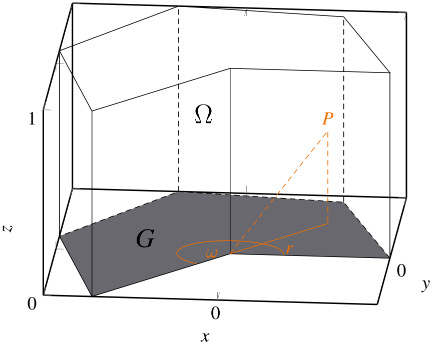
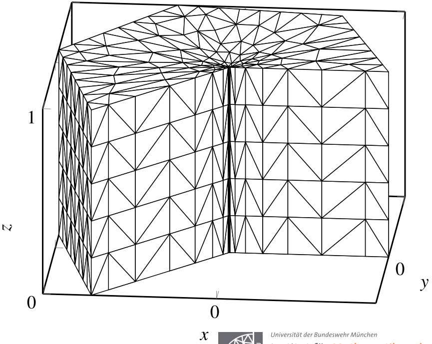
Anisotropic mesh grading

Edge singularities, $\omega > \pi$

- velocity has singular part of type r^λ or $r^{\pi/\omega}$, pressure $r^{\lambda-1}$, $\lambda \in (1/2, 1)$ is smallest positive solution of $\sin(\lambda\omega) = -\lambda \sin(\omega)$
- anisotropic mesh: $h_{z,T} = h$ and $h_{x,T} \sim h_{y,T} \sim hr_T^{1-\mu}$ with $\mu < \lambda$





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Error estimates on anisotropic mesh, edge singularity

Estimate for the modified Crouzeix–Raviart method[6]

Let the mesh be refined in the structured way according to $\mu < \lambda$. Then we have the estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} \leq \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h^0} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,h} + \frac{1}{\nu} \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h^0} \frac{|(\nabla_h \boldsymbol{u}, \nabla_h \boldsymbol{w}_h) - (\boldsymbol{f}, I_h \boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_{1,h}} \leq ch \frac{1}{\nu} \|\mathbb{P}\boldsymbol{f}\|_0$$

where $\mathbb{P}$ is the Helmholtz projector, $\boldsymbol{f} = \mathbb{P}\boldsymbol{f} + \nabla\phi$.

Estimate for the unmodified Crouzeix–Raviart method[7]

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,h} \leq c \left[\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{1,h} + \frac{1}{\nu} \left(\inf_{q_h \in Q_h} \|p - q_h\|_0 + h\|\boldsymbol{f}\|_0 \right) \right] \leq ch \frac{1}{\nu} \|\boldsymbol{f}\|_0.$$

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[6] T. Apel and V. Kempf. "Pressure-robust error estimate of optimal order for the Stokes equations: domains with re-entrant edges and anisotropic mesh grading". *Calcolo* 58.2 (2021), 15

[7] T. Apel, S. Nicaise, and J. Schöberl. "A non-conforming finite element method with anisotropic mesh grading for the Stokes problem in domains with edges". *IMA J. Numer. Anal.* 21.4 (2001), 843–856

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4. Discretization error estimate for CR with reconstruction
5. Numerical example

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Singular edge

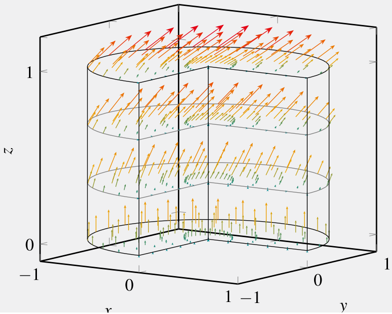
Manufactured solution

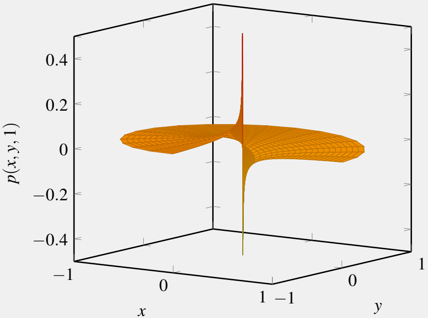
$\Omega = \{(r\cos(\varphi), r\sin(\varphi), z) \in \mathbb{R}^3 : 0 < r < 1, 0 < \varphi < \frac{3\pi}{2}, 0 < z < 1\}$, $\lambda \approx 0.5445$

$$\boldsymbol{u} = \frac{1}{\nu} \begin{pmatrix} zr^{\lambda}\Phi_1(\varphi) \\ zr^{\lambda}\Phi_2(\varphi) \\ r^{2/3}\sin\left(\frac{2}{3}\varphi\right) \end{pmatrix} \qquad \boldsymbol{f}_i = \begin{pmatrix} 0 \\ 0 \\ r^{\lambda-1}\Phi_p(\varphi) \end{pmatrix} + \nabla\psi_i, \quad i = 1, 2$$

$$p = 2\lambda z r^{\lambda-1}\Phi_p(\varphi) + \psi_i \qquad \psi_1 = 0, \quad \psi_2 = 10 r^{\lambda}\Phi_p(\varphi)$$

Solution for $i = 1$





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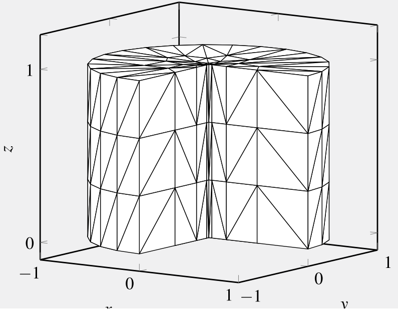
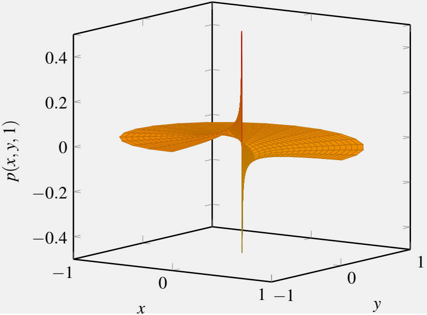
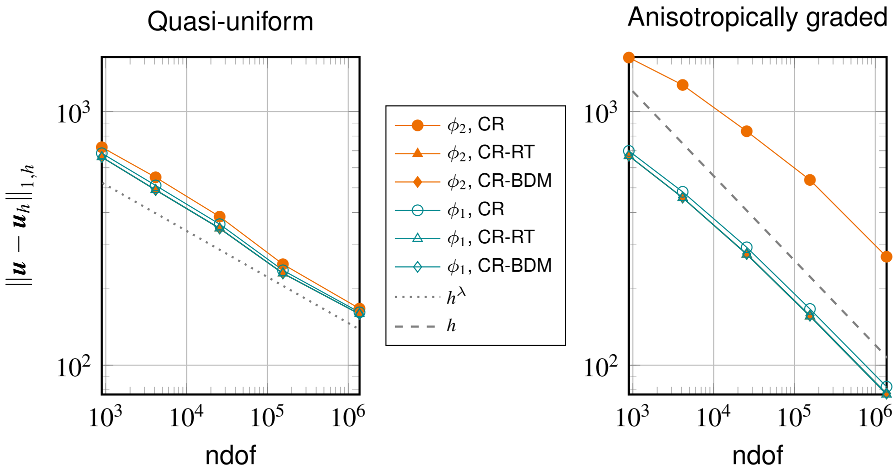
Anisotropic, graded mesh





Quasi-uniform and graded meshes, $\nu = 10^{-3}$



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Summary

New results

- anisotropic interpolation error estimates for BDM interpolation[8],
- pressure robust a-priori estimate for modified Crouzeix–Raviart method for regular solution[9] and
- on domains with re-entrant edges[10]

[8] T. Apel and V. Kempf. “Brezzi–Douglas–Marini interpolation of any order on anisotropic triangles and tetrahedra”. *SIAM J. Numer. Anal.* 58.3 (2020), 1696–1718

[9] T. Apel, V. Kempf, A. Linke, and C. Merdon. “A nonconforming pressure-robust finite element method for the Stokes equations on anisotropic meshes”. *IMA J. Numer. Anal.* (2021)

[10] T. Apel and V. Kempf. “Pressure-robust error estimate of optimal order for the Stokes equations: domains with re-entrant edges and anisotropic mesh grading”. *Calcolo* 58.2 (2021), 15

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