Error Estimation for Stochastic Galerkin Finite Element Approximations

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Outline

Part I: Parameter-dependent Elliptic PDEs

- Stochastic Galerkin Approximation: The Basics
- Error Estimation Strategy

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- ▷ Stochastic Galerkin Approximation: The Basics
- ▷ Error Estimation Strategy

Part II: Parameter-dependent Elasticity Problems

Collaborators: Arbaz Khan (IIT Roorkee), David Silvester (Manchester)

- ▷ Stochastic Galerkin Mixed FEM
- ▷ Error Estimation & Adaptivity

Forward UQ: PDEs + Uncertain Inputs

Aim: Propagate **uncertainty** from model inputs to outputs

$$-\nabla \cdot (a(\mathbf{x}, \mathbf{y}) \nabla u(\mathbf{x}, \mathbf{y})) = f(\mathbf{x}).$$

- Represent uncertain inputs as **random variables/parameters y**.
- Approximate QoIs related to the solution of the model.

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Common methods for **forward UQ**:

- Monte Carlo methods
- **Stochastic Galerkin → not a sampling method!**
- Stochastic collocation
- ...

(Stochastic) Galerkin Approximation

▷ **Parametric PDE:** Find $u : D \times \Gamma \to \mathbb{R}$ such that

$$-\nabla \cdot (a(\mathbf{x}, \mathbf{y}) \nabla u(\mathbf{x}, \mathbf{y})) = f(\mathbf{x}), \qquad \mathbf{x} \in D \subset \mathbb{R}^d, \quad \mathbf{y} \in \Gamma.$$

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$$B(u, v) = \ell(v) \qquad \forall v \in V.$$

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- **Linear System:** $A\mathbf{u} = \mathbf{f}$

- **Error:** The error $e := u - u_{\mathrm{gal}} \in V$ satisfies:

$$B(e, v) = \ell(v) - B(u_{\mathrm{gal}}, v) \qquad \forall\, v \in V.$$

Exploiting Structure

- **Stochastically linear** inputs: Assume (in this talk):

$$a(\mathbf{x}, \mathbf{y}) = a_0(\mathbf{x}) + \sum_{m=1}^{\infty} a_m(\mathbf{x}) y_m$$

$y_m = \xi_m(\omega)$ are images of independent & bounded random variables.

- **Tensor Product** approximation spaces: $\widehat{V} = X \otimes \mathcal{P}$

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For this simple test problem,

$$V := L^2_\pi(\Gamma, H^1_0(D)) \cong H^1_0(D) \otimes L^2_\pi(\Gamma)$$

where π is a **probability measure**, so we choose

$$X \subset H^1_0(D), \qquad \mathcal{P} \subset L^2_\pi(\Gamma).$$

Useful as a 'Surrogate'

$$u_{\mathrm{gal}} \in \widehat{V} = X \otimes \mathcal{P}$$

- $X = \mathrm{span}\,\{\phi_i(\mathbf{x}), i = 1, \ldots, n_X\}$ is a **FEM** space on D.
- $\mathcal{P} = \mathrm{span}\,\{\psi_{\boldsymbol{\alpha}}(\mathbf{y}), \boldsymbol{\alpha} \in \Lambda\}$ is a set of **global polynomials** on Γ and

$$\int_{\Gamma} \psi_{\boldsymbol{\alpha}}(\mathbf{y})\psi_{\boldsymbol{\beta}}(\mathbf{y})\, d\pi(\mathbf{y}) = 0 \qquad \text{when} \quad \boldsymbol{\alpha} \neq \boldsymbol{\beta}.$$

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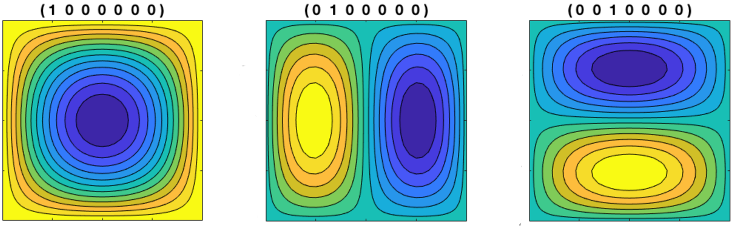
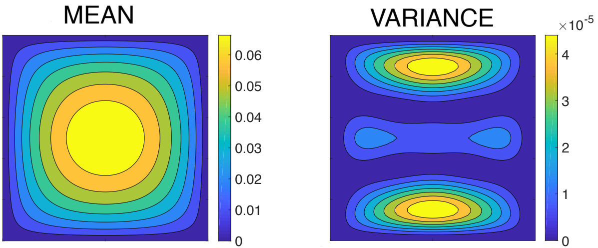
- Λ is a set of **multi-indices** $\boldsymbol{\alpha}$ with a finite no of non-zero entries.

Solving $A\mathbf{u} = \mathbf{f}$ gives coefficients $u_{i,\boldsymbol{\alpha}}$ such that

$$u_{\rm gal}(\mathbf{x}, \mathbf{y}) = \sum_{\boldsymbol{\alpha} \in \Lambda} \left(\sum_{i=1}^{n_X} u_{i,\boldsymbol{\alpha}} \phi_i(\mathbf{x}) \right) \psi_{\boldsymbol{\alpha}}(\mathbf{y}) = \sum_{\boldsymbol{\alpha} \in \Lambda} u_{\boldsymbol{\alpha}}(\mathbf{x}) \psi_{\boldsymbol{\alpha}}(\mathbf{y}).$$

Selected solution modes $u_{\alpha}(\mathbf{x})$ for a test problem with $D = [-1, 1]^2$.





Error Estimation I (An Old Idea)

Starting Point: The true error $e := u - u_{\mathrm{gal}} \in V$ satisfies:

$$B(e, v) = \underbrace{F(v) - B(u_{\mathrm{gal}}, v)}_{\text{residual } R(v)} \qquad \forall\, v \in V.$$

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To reduce costs, we approximate e^* by $e_{\mathrm{approx}} \in V_{\mathrm{new}}$ satisfying

$$B_0(e_{\mathrm{approx}}, v) = R(v) \qquad \forall\, v \in V_{\mathrm{new}},$$

where $B_0(\cdot,\cdot) \approx B(\cdot,\cdot)$ and define $\eta := \|e_{\mathrm{approx}}\|_{B_0}$.

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where $B_0(\cdot,\cdot) \approx B(\cdot,\cdot)$ and define $\eta := \|e_{\rm approx}\|_{B_0}$.

One can then prove **two-sided bounds**:

$$C_1\, \eta \le \|u - u_{\rm gal}\|_B \le C_2\, \eta.$$

Error Estimation II

Here, a natural choice is the **'mean part'** of $B(\cdot, \cdot)$:

$$B_0(u, v) \quad := \quad \int_\Gamma \int_D a_0(\mathbf{x}) \nabla u(\mathbf{x}, \mathbf{y}) \cdot \nabla v(\mathbf{x}, \mathbf{y}) \; d\mathbf{x} d\pi(\mathbf{y}).$$

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Choose Y (FEM space), $\mathcal{Q}$ (polynomials in subset of y_m) with

$$X \cap Y = \{0\}, \qquad \mathcal{P} \cap \mathcal{Q} = \{0\}$$

and define

$$V_{\mathrm{new}} := \underbrace{(Y \otimes \mathcal{P})}_{V_{Y\mathcal{P}}} \oplus \underbrace{(X \otimes \mathcal{Q})}_{V_{X\mathcal{Q}}}$$

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With these choices $e_{\mathrm{approx}} = e_{Y\mathcal{P}} + e_{X\mathcal{Q}}$ where:

1. $e_{Y\mathcal{P}} \in V_{Y\mathcal{P}}$ satisfies $B_0(e_{Y\mathcal{P}}, v) = R(v) \quad \forall\, v \in V_{Y\mathcal{P}}$,
2. $e_{X\mathcal{Q}} \in V_{X\mathcal{Q}}$ satisfies $B_0(e_{X\mathcal{Q}}, v) = R(v) \quad \forall\, v \in V_{X\mathcal{Q}}$.

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Problem 1 can be broken up using **'element residual technique'**.

Error Estimation III

$$\eta := \|e_{\mathrm{approx}}\|_{B_0} = \left(\|e_{Y\mathcal{P}}\|_{B_0}^2 + \|e_{X\mathcal{Q}}\|_{B_0}^2\right)^{1/2},$$

$\|e_{Y\mathcal{P}}\|_{B_0}$ and $\|e_{X\mathcal{Q}}\|_{B_0}$ are estimators for the **error reduction** that would be achieved by computing one of two new approximations:

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1. $u_{\rm new} \in (X \oplus Y) \otimes \mathcal{P}$ **(spatial refinement)**

$$C_1 \, \|e_{Y\mathcal{P}}\|_{B_0} \le \|u_{new} - u_{\rm gal}\|_B \le C_3 \, \|e_{Y\mathcal{P}}\|_{B_0}$$

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2. $u_{\mathrm{new}} \in X \otimes (\mathcal{P} \oplus \mathcal{Q})$ **(parametric enrichment)**

$$C_1 \|e_{X\mathcal{Q}}\|_{B_0} \leq \|u_{new} - u_{\mathrm{gal}}\|_B \leq C_4 \|e_{X\mathcal{Q}}\|_{B_0}$$

▷ This is the starting point for **adaptivity** ...

Questions?

References:

- **S-IFISS (MATLAB Toolbox):**

  http : //www.manchester.ac.uk/ifiss/sifiss.html

- Efficient adaptive **multilevel** stochastic Galerkin approximation using implicit a posteriori error estimation. **A. Crowder**, C.E. Powell, SISC, 41(3), 2019.

- Energy norm a posteriori error estimation for parametric operator equations, **A. Bespalov**, C.E. Powell, D. Silvester, SISC. 36(2), 2014.

Linear Elasticity (Herrmann Formulation)

Find $\mathbf{u} : D \to \mathbb{R}^d$ and $p : D \to \mathbb{R}$ such that

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$$
$$\nabla \cdot \mathbf{u}(\mathbf{x}) + \frac{1}{\lambda} p(\mathbf{x}) = \mathbf{0}$$

(+ boundary conditions), where

$$\boldsymbol{\sigma} := 2\mu\, \epsilon(\mathbf{u}) - p\mathbf{I}, \quad \epsilon(\mathbf{u}) := (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)/2$$

and the Lamé coefficients are:

$$\mu(\mathbf{x}) = \frac{E(\mathbf{x})}{2(1+\nu)}, \quad \lambda(\mathbf{x}) = \frac{E(\mathbf{x})\,\nu}{(1+\nu)(1-2\nu)}.$$

Incompressible Case: As the **Poisson ratio** $\nu \to 1/2$, then $\lambda \to \infty$.

Parameter-Dependent Young's Modulus

$$E(\mathbf{x},\mathbf{y}) := e_0(\mathbf{x}) + \sum_{m=1}^{\infty} e_m(\mathbf{x})\, y_m, \qquad \mathbf{x} \in D,\ \mathbf{y} \in \Gamma.$$

Parametric PDE: Find $\mathbf{u} : D \times \Gamma \to \mathbb{R}^2$ and $p : D \times \Gamma \to \mathbb{R}$ such that

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Stochastically Linear Three-field Formulation

Issue: When the inputs depend on the parameters y_m in a **non-linear** way, the SGFEM system matrix is **block dense**.

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Introducing the **'modified pressure'**

$$\tilde{p} = \frac{1}{E} p := -\underbrace{\frac{\lambda}{E}}_{=:\tilde{\lambda}} \nabla \cdot \mathbf{u}$$

gives a parametric **three-field formulation**:

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma} &= \mathbf{f} \\ \nabla \cdot \mathbf{u} + \frac{1}{\tilde{\lambda}} \tilde{p} &= 0 \\ \frac{1}{\tilde{\lambda}} p - \frac{E}{\tilde{\lambda}} \tilde{p} &= 0 \end{aligned}$$

in which E but **not** E^{-1} appears.

$$E(\mathbf{x},\mathbf{y}) := e_0(\mathbf{x}) + \sum_{m=1}^{\infty} e_m(\mathbf{x})\, y_m, \qquad \mathbf{x} \in D,\ \mathbf{y} \in \Gamma.$$

(stochastically linear) with standard assumptions

- $0 < E_{\min} \leq E(\mathbf{x},\mathbf{y}) \leq E_{\max} < \infty \quad \text{a.e. in } D \times \Gamma.$
- $0 < e_0^{\min} \leq e_0(\mathbf{x}) \leq e_0^{\max} < \infty \quad \text{a.e. in } D.$

Weak formulation requires the **Bochner spaces:**

$$\mathbf{V} := L^2_\pi(\Gamma, \mathbf{H}^1_0(D)), \quad W := L^2_\pi(\Gamma, L^2(D)).$$

Weak Formulation

Find $(\mathbf{u}, p, \tilde{p}) \in \mathbf{V} \times W \times W$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= f(\mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) - c(\tilde{p}, q) &= 0 && \forall q \in W, \\ -c(p, \tilde{q}) + d(\tilde{p}, \tilde{q}) &= 0 && \forall \tilde{q} \in W, \end{aligned}$$

where

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\end{aligned}
$$

where

$$
\begin{aligned}
a(\mathbf{u}, \mathbf{v}) &:= \alpha \int_\Gamma \int_D E(\mathbf{x}, \mathbf{y})\, \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\, d\mathbf{x} d\pi(\mathbf{y}), \\
b(\mathbf{v}, p) &:= -\int_\Gamma \int_D p \nabla \cdot \mathbf{v}\, d\mathbf{x} d\pi(\mathbf{y}), \\
c(p, q) &:= \tilde{\lambda}^{-1} \int_\Gamma \int_D pq\, d\mathbf{x} d\pi(\mathbf{y}), \\
d(p, q) &:= \tilde{\lambda}^{-1} \int_\Gamma \int_D E(\mathbf{x}, \mathbf{y}) pq\, d\mathbf{x} d\pi(\mathbf{y}),
\end{aligned}
$$

and

$$\alpha := \frac{1}{(1+\nu)}$$

(**Note:** $\tilde{\lambda}^{-1} \to 0$ as $\nu \to 1/2$).

A unique solution $(\mathbf{u}, p, \tilde{p}) \in \mathbf{V} \times W \times W$ exists satisfying

$$|||(\mathbf{u}, p, \tilde{p})||| \leq \underbrace{(C/E_{\min})\, \alpha^{-1/2}}_{\text{bounded as } \nu \to 1/2} \, ||\mathbf{f}||_{L^2(D)}$$

where C depends on $E_{\max}$ and $|||\cdot|||$ is a **weighted norm**

$$|||(\mathbf{u}, p, \tilde{p})|||^2 := \alpha \, ||\nabla \mathbf{u}||^2_{\mathbf{W}} + (\alpha^{-1} + \tilde{\lambda}^{-1})||p||^2_W + \tilde{\lambda}^{-1}||\tilde{p}||^2_W$$

where

$$\| \cdot \|_{\mathbf{W}} := \| \cdot \|_{L^2_\pi(\Gamma,(L^2(D))^{d\times d})}, \qquad \| \cdot \|_W := \| \cdot \|_{L^2_\pi(\Gamma, L^2(D))}.$$

Stochastic Galerkin Mixed FEM (SGMFEM)

- $\mathbf{V}_h \subset \mathbf{H}_0^1(D)$, $W_h \subset L^2(D)$ (**inf-sup stable** FEM pair).

$$\sup_{0 \neq \mathbf{v} \in \mathbf{V}_h} \frac{\int_D q \, \nabla \cdot \mathbf{v}}{||\nabla \mathbf{v}||_{\mathbf{L}^2(D)}} \geq \beta \, ||q||_{L^2(D)} \quad \forall q \in W_h$$

Examples: $\mathbf{Q}_2$-Q_1, $\mathbf{P}_2$-P_1, ...

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- Define $\mathcal{P} = \mathrm{span}\{\psi_{\boldsymbol{\alpha}}(\mathbf{y}), \boldsymbol{\alpha} \in \Lambda\}$ as before.

- Pair of inf-sup stable SGMFEM approximation spaces:

$$\widehat{\mathbf{V}} := \mathbf{V}_h \otimes \mathcal{P} \quad \text{and} \quad \widehat{W} := W_h \otimes \mathcal{P}.$$

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The associated discrete problem has a unique solution that is also bounded w.r.t $|||\cdot|||$ as the Poisson ratio $\nu \to 1/2$.

Error estimation I

The current SGMFEM approximation satisfies

$$\mathbf{u}_{\rm gal} \in \widehat{\mathbf{V}} := \mathbf{V}_h \otimes \mathcal{P}, \qquad p_{\rm gal},\ \tilde{p}_{\rm gal} \in \widehat{W} := W_h \otimes \mathcal{P}.$$

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Error equations

Substituting $\mathbf{u} = \mathbf{u}_{\text{gal}} + e^{\mathbf{u}}$, $p = p_{\text{gal}} + e^{p}$ and $\tilde{p} = \tilde{p}_{\text{gal}} + e^{\tilde{p}}$ gives:

$$\begin{aligned} a(e^{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, e^{p}) &= \mathcal{R}^{\mathbf{u}}(\mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}, \\ b(e^{\mathbf{u}}, q) - c(e^{\tilde{p}}, q) &= \mathcal{R}^{p}(q) \quad \forall q \in \mathcal{W}, \\ -c(e^{p}, \tilde{q}) + d(e^{\tilde{p}}, \tilde{q}) &= \mathcal{R}^{\tilde{p}}(\tilde{q}) \quad \forall \tilde{q} \in \mathcal{W}, \end{aligned}$$

where $\mathcal{R}^{\mathbf{u}}(\mathbf{v})$, $\mathcal{R}^{p}(q)$ and $\mathcal{R}^{\tilde{p}}(\tilde{q})$ are the residuals.

Approximate $e^{\mathbf{u}}$, e^{p}, $e^{\tilde{p}}$ in spaces that are **richer** than $\widehat{\mathbf{V}}$ and $\widehat{W}$.

Hierarchical Approach

- Choose FEM detail spaces $\widetilde{\mathbf{V}}_h \subset \mathbf{H}_0^1(D)$, $\widetilde{W}_h \subset L^2(D)$ with

$$\mathbf{V}_h \cap \widetilde{\mathbf{V}}_h = \{\mathbf{0}\}, \qquad W_h \cap \widetilde{W}_h = \{0\},$$

such that the enriched spaces are an **inf-sup stable** pair:

$$\mathbf{V}_h^* := \mathbf{V}_h \oplus \widetilde{\mathbf{V}}_h \quad \text{and} \quad W_h^* := W_h \oplus \widetilde{W}_h.$$

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- Error approximation spaces:

$$\mathbf{V}^* := \widehat{\mathbf{V}} \oplus \left(\widetilde{\mathbf{V}}_h \otimes \mathcal{P}\right) \oplus (\mathbf{V}_h \otimes \mathcal{Q}),$$
$$W^* := \widehat{W} \oplus \left(\widetilde{W}_h \otimes \mathcal{P}\right) \oplus (W_h \otimes \mathcal{Q}).$$

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Error Estimation II

Replace bilinear forms on left-hand side of error equations with **simpler ones** so that resulting problem **decouples**.

Simplified Error Equations

Find $\mathbf{e}^{\mathbf{u}}_{\text{approx}} \in \mathbf{V}_{\text{new}}$, $e^{p}_{\text{approx}} \in W_{\text{new}}$ and $e^{\tilde{p}}_{\text{approx}} \in W_{\text{new}}$ such that:

$$
\begin{aligned}
a_0(\mathbf{e}^{\mathbf{u}}_{\text{approx}}, \mathbf{v}) &:= \mathcal{R}^{\mathbf{u}}(\mathbf{v}), && \forall \mathbf{v} \in \mathbf{V}_{\text{new}} \\
c_0(e^{p}_{\text{approx}}, q) &:= \mathcal{R}^{p}(q), && \forall q \in W_{\text{new}}, \\
d_0(e^{\tilde{p}}_{\text{approx}}, \tilde{q}) &:= \mathcal{R}^{\tilde{p}}(\tilde{q}), && \forall \tilde{q} \in W_{\text{new}}.
\end{aligned}
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The global **a posteriori error estimate** is defined as

$$\eta := (\eta_{\mathbf{u}}^2 + \eta_p^2 + \eta_{\tilde{p}}^2)^{1/2},$$

where

$$\eta_{\mathbf{u}} := \|\mathbf{e}^{\mathbf{u}}_{\text{approx}}\|_{a_0}, \quad \eta_p := \|e^{p}_{\text{approx}}\|_{c_0}, \quad \eta_{\tilde{p}} := \|e^{\tilde{p}}_{\text{approx}}\|_{d_0}.$$

Error Estimation III

$$\eta := (\eta_{\mathbf{u}}^2 + \eta_p^2 + \eta_{\tilde{p}}^2)^{1/2}.$$

Two-sided error bounds

$$C_1\,\eta \le |||(e^{\mathbf{u}}, e^{p}, e^{\tilde{p}})||| \le C_2\,\eta,$$

where C_1, C_2 are **independent** of the **discretization** parameters and ν.

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where C_1, C_2 are **independent** of the **discretization** parameters and ν.

Exploiting the structure of $\mathbf{V}_{\text{new}}, W_{\text{new}}$, the estimator can be **decomposed**.

$$\eta_{\mathbf{u}}^2 = \|\mathbf{e}^{\mathbf{u}}_{\text{approx}}\|^2_{a_0} = \|\mathbf{e}^{\mathbf{u}}_{\text{spatial}}\|^2_{a_0} + \|\mathbf{e}^{\mathbf{u}}_{\text{param}}\|^2_{a_0}$$

where

$$\mathbf{e}^{\mathbf{u}}_{\text{spatial}} \in \widetilde{\mathbf{V}}_h \otimes \mathcal{P}, \qquad \mathbf{e}^{\mathbf{u}}_{\text{param}} \in \mathbf{V}_h \otimes \mathcal{Q}$$

and similarly for η_p^2 and $\eta_{\tilde{p}}^2$.

Adaptivity

Use contributions to η as indicators for the **error reduction** that would be achieved by enriching either (i) $\mathbf{V}_h$-W_h or (ii) $\mathcal{P}$ at the next step.

E.g., define a **spatial** error reduction indicator $\eta_{\rm spatial}$ via

$$\eta^2_{\rm spatial} := \|\mathbf{e}^{\mathbf{u}}_{\rm spatial}\|^2_{a_0} + \|e^{p}_{\rm spatial}\|^2_{c_0} + \|e^{\tilde{p}}_{\rm spatial}\|^2_{d_0}.$$

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E.g., define a **spatial** error reduction indicator η_{spatial} via

$$\eta_{\mathrm{spatial}}^2 := \|\mathbf{e}_{\mathrm{spatial}}^{\mathbf{u}}\|_{a_0}^2 + \|e_{\mathrm{spatial}}^{p}\|_{c_0}^2 + \|e_{\mathrm{spatial}}^{\tilde{p}}\|_{d_0}^2.$$

Let $\mathbf{u}_{\mathrm{new}}, p_{\mathrm{new}}, \tilde{p}_{\mathrm{new}}$ be the SGFEM approximation associated with

$$\left(\mathbf{V}_h \oplus \widetilde{\mathbf{V}}_h\right) \otimes \mathcal{P}, \qquad \left(W_h \oplus \widetilde{W}_h\right) \otimes \mathcal{P}.$$

Spatial Error Reduction Indicator

$$C_1\, \eta_{\mathrm{spatial}} \le |||(\mathbf{u}_{\mathrm{new}} - \mathbf{u}_{\mathrm{gal}}, p_{\mathrm{new}} - p_{\mathrm{gal}}, \tilde{p}_{\mathrm{new}} - \tilde{p}_{\mathrm{gal}})||| \le C_3\, \eta_{\mathrm{spatial}}$$

Numerical Example

Test problem with **spatial singularities** (which become weaker as $\nu \to \frac{1}{2}$).

- Spatial domain: $D = (0,1) \times (0,1)$.
- Mixed bc's: $\boldsymbol{\sigma}\mathbf{n} = \mathbf{0}$ on right-hand edge and $\mathbf{u} = \mathbf{0}$ on $\partial D \setminus \partial D_N$.
- Body force: $\mathbf{f} = (0.1, 0)^\top$.
- Parameter-dependent Young's modulus:

$$E(\mathbf{x}, \mathbf{y}) = 1 + \sum_{m=1}^{\infty} a_m(\mathbf{x})\, y_m, \qquad \|a_m(\mathbf{x})\|_\infty \sim m^{-2}, \quad y_m \sim U(-1,1).$$

- Mixed FEM approximation : $\mathbf{V}_h$–W_h = $\mathbf{P}_2$–P_1 (triangles).

Non-adaptive: Poisson ratio $\nu = 0.4$

Galerkin approximation using $\mathbf{P}_2$–P_1 on a uniform FEM mesh, with $\mathcal{P}$ the set of polynomials of total degree ≤ 4 in $M = 8$ parameters.

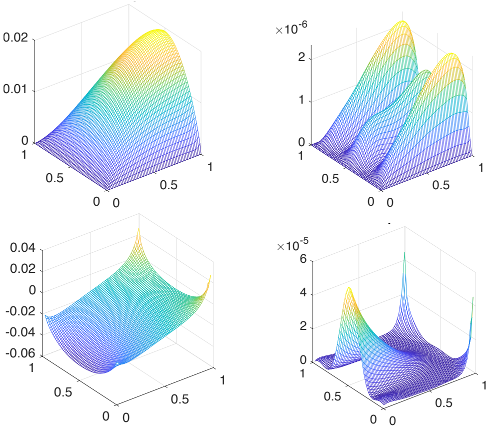
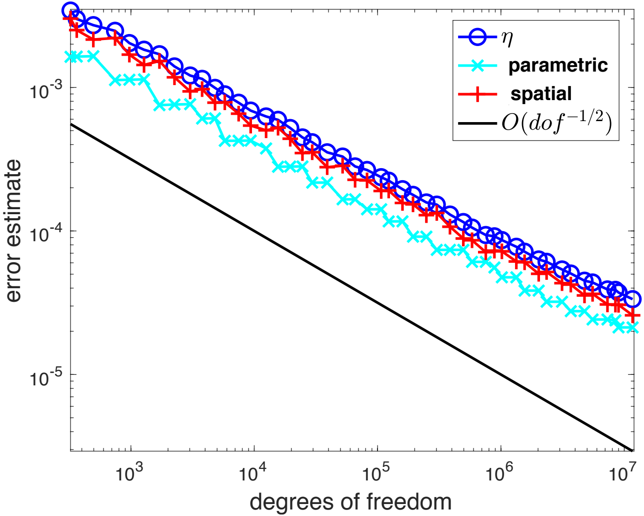
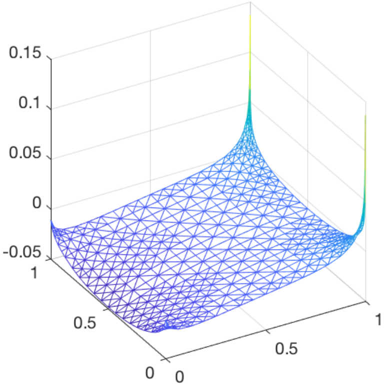


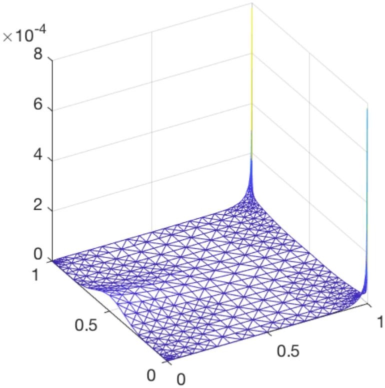
Figure: Top: Estimated mean (left) and variance (right) of u_x. Bottom: Estimated mean (left) and variance (right) of p.

At each step choose between (i) local FEM mesh refinement, or (ii) parametric enrichment.



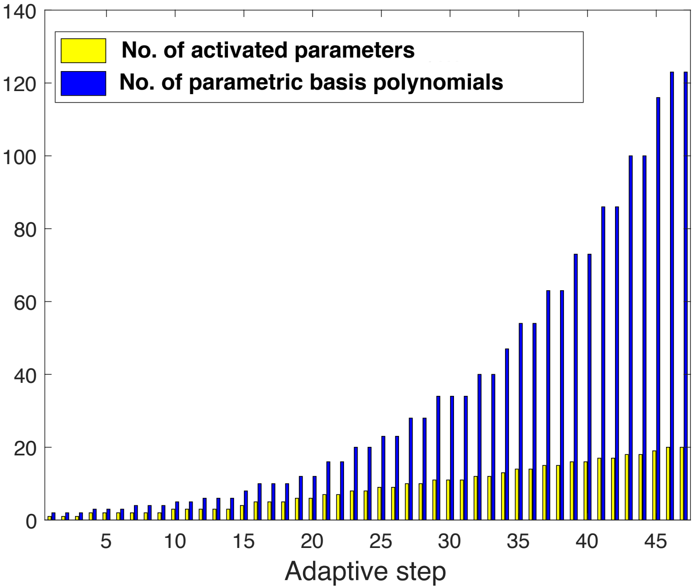
Improved Pressure Approximation ($\nu = 0.4$)



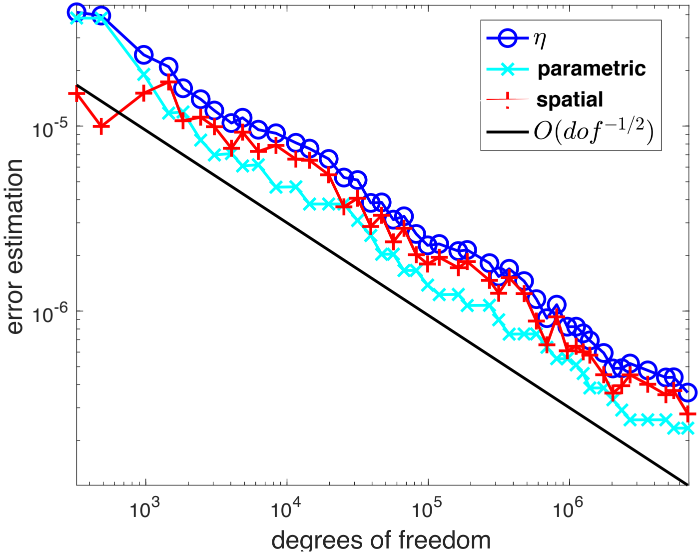


Mean $\mathbb{E}[p_{\rm gal}]$ (left) and **variance** $\mathbb{V}[p_{\rm gal}]$ (right).

Adaptive: Activated Indices ($\nu = 0.4$)



Adaptive: Poisson Ratio $\nu = 0.49999$



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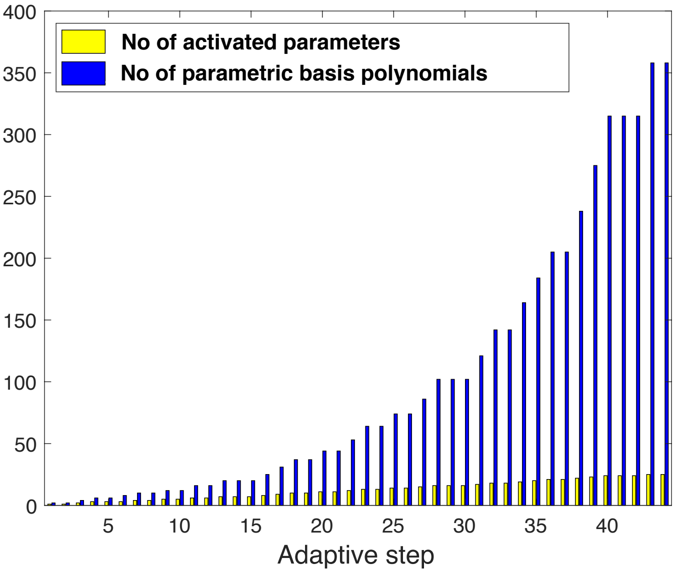
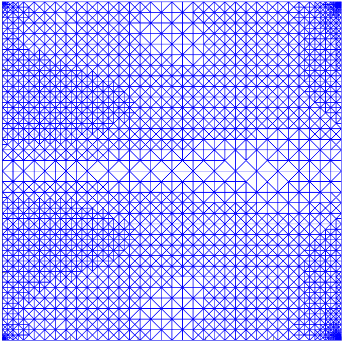
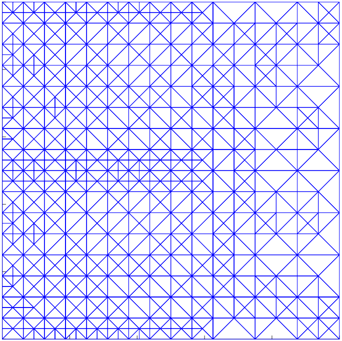


Adaptive FEM Meshes





Poisson ratio: $\nu = 0.4$ (right) and $\nu = 0.49999$ (right).

Summary: Linear Elasticity Problem

- **Two-sided bounds** for error and estimates of potential error reduction.
- **Adaptive** SGFEM algorithm.
- Error estimation & solver both robust for **nearly incompressible** case.

References

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- Khan et al., Robust **preconditioning** for stochastic Galerkin formulations of parameter-dept. nearly incompressible elasticity equations, **SISC 41(1)**, 2019.