The Stokes equations with singular boundary data.

Ricardo G. Durán

Department of Mathematics
University of Buenos Aires
Luis A. Santalo Institute, UBA-CONICET

Dublin
April 22, 2021

Joint work with Lucia Gastaldi and Ariel Lombardi

ABSTRACT

- The Stokes equations.
- Existence and uniqueness.
- Equivalent forms of the inf-sup condition.
- Bogovskii's solution of the divergence.
- Singular Dirichlet data.
- Numerical approximation by regularization and error estimates.
- Numerical results.

THE STOKES EQUATIONS

They model the displacement of a viscous incompressible fluid contained in a domain Ω when the non-linear convection terms can be neglected (i. e. small Reynold's numbers)

In the stationary case and assuming homogeneous Dirichlet conditions they can be written as,

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases}$$

They model the displacement of a viscous incompressible fluid contained in a domain Ω when the non-linear convection terms can be neglected (i. e. small Reynold's numbers)

In the stationary case and assuming homogeneous Dirichlet conditions they can be written as,

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega \end{cases}$$

This is the usual form in Finite Element books and papers.

However, it is important to mention that in order to obtain the physical natural boundary conditions in the integration by parts the correct formulation is

$$\begin{cases} -2\mu \mathrm{Div}\, \varepsilon(\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega \\ \mathrm{div}\, \mathbf{u} = 0 & \text{in } \Omega \end{cases}$$

where

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Therefore this form has to be used when dealing with Neumann type boundary conditions.

An important difficult that arise when one works with this form is the necessity of the Korn inequality.

For simplicity we will work with Dirichlet boundary conditions.

SOBOLEV SPACES

Given a domain $\Omega \subset \mathbb{R}^n$ we use the standard notation:

$$H^1(\Omega) = \left\{ f \in L^2(\Omega) \, : \, |\nabla f| \in L^2(\Omega) \right\}$$

$$H_0^1(\Omega) = \overline{C_0^\infty(\Omega)}$$

and in general, for $k \in \mathbb{N}$,

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) \, : \, D^\alpha f \in L^2(\Omega), \forall |\alpha| \leq k \right\}$$

$$\|f\|_{k,\Omega} = \|f\|_{H^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|^2 \right)^{1/2} \quad , \quad \|f\|_{0,\Omega} = \|f\|_{L^2(\Omega)}$$

For $D = \Omega$ or $D = \Gamma = \partial\Omega$ (or subsets of them), if $d = dim\, D$ the fractional Sobolev or Besov spaces are defined, for $0 < s < 1$, as

$$H^s(D) = \left\{ f \in L^2(D) \,:\, |f|_s < \infty \right\}$$

where

$$|f|^2_{H^s(D)} = |f|^2_{s,D} := \int_D \int_D \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}} dxdy$$

$H^s(D)$ is a Hilbert space with norm

$$\|f\|^2_{s,D} = \|f\|^2_{L^2(D)} + |f|^2_{s,D}$$

NEGATIVE ORDER SPACES

We will also work with the dual spaces:

On Γ, $0 < s < 1$,

$$H^{-s}(\Gamma) = H^s(\Gamma)'$$

and on Ω,

$$H^{-1}(\Omega) = H_0^1(\Omega)'$$

and will use the standard notation

$$L_0^2(\Omega) = \left\{ f \in L^2(\Omega) \; : \; \int_\Omega f = 0 \right\} \simeq L^2(\Omega)/\mathbb{R}$$

TRACE THEOREMS

A particular case of the fractional norms was introduced by Gagliardo to characterize the restrictions to Γ of Sobolev functions.

If Ω is Lipschitz, the restriction to the boundary of any $u \in H^1(\Omega)$ is well defined and belongs to $H^{1/2}(\Gamma)$. Moreover,

$$\|u|_\Gamma\|_{H^{1/2}(\Gamma)} \leq C\|u\|_{H^1(\Omega)}$$

And conversely, given any $f \in H^{1/2}(\Gamma)$, there exists $u \in H^1(\Omega)$ such that

$$f = u|_\Gamma \qquad y \quad \|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\Gamma)}$$

WELL POSEDNESS OF STOKES EQUATIONS

We consider the subspace of divergence free vector fields:

$$Ker(\operatorname{div}) = \{\mathbf{u} \in H_0^1(\Omega)^n : \operatorname{div} \mathbf{u} = 0\}$$

Integrating by parts and using that ∇p is orthogonal to $Ker(\operatorname{div})$ we obtain the weak formulation

$$\mu \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} = \int_\Omega \mathbf{f}\mathbf{v} \quad \forall \mathbf{v} \in Ker(\operatorname{div})$$

Then, by the Lax-Milgram theorem there exists a unique solution $\mathbf{u} \in Ker(\operatorname{div})$, and moreover

$$\|\mathbf{u}\|_{H_0^1(\Omega)} \le C\|\mathbf{f}\|_{H^{-1}(\Omega)}.$$

WELL POSEDNESS OF STOKES EQUATIONS

We consider the subspace of divergence free vector fields:

$$Ker(\text{div}) = \{\mathbf{u} \in H_0^1(\Omega)^n : \text{div}\,\mathbf{u} = 0\}$$

Integrating by parts and using that ∇p is orthogonal to $Ker(\text{div})$ we obtain the weak formulation

$$\mu \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} = \int_\Omega \mathbf{f}\mathbf{v} \quad \forall \mathbf{v} \in Ker(\text{div})$$

Then, by the Lax-Milgram theorem there exists a unique solution $\mathbf{u} \in Ker(\text{div})$, and moreover

$$\|\mathbf{u}\|_{H_0^1(\Omega)} \le C\|\mathbf{f}\|_{H^{-1}(\Omega)}.$$

It remains to show the existence of $p \in L_0^2(\Omega)$ such that,

$$\nabla p = \mathbf{f} + \Delta \mathbf{u}.$$

A classic result in Functional Analysis says that, for A and A^* adjoint operators,

$$(Ker\, A)^{\perp} = \overline{Im\, A^*}$$

But

$$\mathbf{f} + \Delta \mathbf{u} \in (Ker(\mathrm{div}\,))^{\perp}$$

and

$$\mathrm{div}\, : H_0^1(\Omega)^n \longrightarrow L_0^2(\Omega) \quad \text{and} \quad \nabla : L_0^2(\Omega) \longrightarrow H^{-1}(\Omega)^n$$

are adjoint operators.

Therefore,

$$\mathbf{f} + \Delta \mathbf{u} \in \overline{Im\, \nabla} = (Ker(\mathrm{div}\,))^{\perp}$$

Then, the existence of $p \in L^2_0(\Omega)$ such that,

$$\nabla p = \mathbf{f} + \Delta \mathbf{u}.$$

is a consequence of the following fundamental result:

$$Im \nabla \quad \text{is a closed subspace of} \quad H^{-1}(\Omega)^n$$

or equivalently

$$\|p\|_{L^2(\Omega)} \leq C \|\nabla p\|_{H^{-1}(\Omega)} \qquad \forall p \in L^2_0(\Omega)$$

EQUIVALENT FORMS

The last inequality is usually called *“inf-sup” condition* because it can be written as

$$\inf_{p\in L^2_0} \sup_{\mathbf{v}\in H^1_0} \frac{\int_\Omega p \,\mathrm{div}\, \mathbf{v}}{\|p\|_0 \|v\|_1} \geq \beta > 0$$

By standard duality arguments this is equivalent to the following result:

$$\forall f \in L^2_0(\Omega) \qquad \exists \mathbf{u} \in H^1_0(\Omega)^n$$

such that,

$$\mathrm{div}\, \mathbf{u} = f \qquad , \qquad \|\mathbf{u}\|_{H^1_0(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

CONSTRUCTIVE APPROACH

There are many arguments to prove this result. Some of them require smoothness assumptions on $\partial\Omega$.
For star-shaped domains (with respect to a ball $B \subset \Omega$) Bogovskii introduced the following constructive solution of $\operatorname{div} \mathbf{u} = f$:

$$\mathbf{u}(x) = \int_\Omega G(x,y)\, f(y) dy$$

where

$$G(x,y) = \int_0^1 \frac{(x-y)}{t} \omega\left(y + \frac{x-y}{t}\right) \frac{dt}{t^n}$$

where $\omega \in C_0^\infty(B)$, $\int \omega = 1$

The construction is elementary: Fundamental theorem of calculus on segments and regularized average using ω.

CONSTRUCTIVE APPROACH

The difficult part of the proof is to show that

$$\|\mathbf{u}\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

This estimate (as well as its generalization to L^p-based norms) can be proved using the Calderón-Zygmund singular integral operators theory.

The L^2 case can be proved also using the Fourier Transform (D.-2012). In this way better estimates for the constant are obtained.

Bogovskii extends the result to Lipschitz domains using that they are finite union of star-shaped ones.

The construction can be generalized to the class of John domains (D., G.Acosta, M. A. Muschietti, 2006)

NON HOMOGENEOUS DIRICHLET CONDITION

Consider the problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{0} \qquad \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 \qquad \text{in } \Omega \\ \mathbf{u} &= \mathbf{g} \qquad \text{on } \Gamma \end{aligned}$$

with **g** satisfying the compatibility condition

$$\int_\Gamma \mathbf{g} \cdot \mathbf{n} = 0$$

If $\mathbf{g} \in H^{\frac{1}{2}}(\Gamma)^n$, using the Gagliardo trace theorem the problem can be reduced to the one analyzed above.

Consequently there exists a unique soution which satisfies

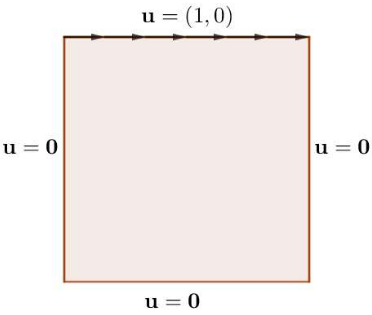
$$\|\mathbf{u}\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq C\|\mathbf{g}\|_{H^{\frac{1}{2}}(\Gamma)}$$

PROBLEM:

$$\mathbf{g} \notin H^{\frac{1}{2}}(\Gamma)^n \qquad \Longrightarrow \qquad \mathbf{u} \notin H^1(\Omega)^n$$

Consequently we cannot apply the classic variational analysis neither for well posedness nor for error estimates.

CLASSIC EXAMPLE: Lid-driven cavity problem



BONDARY DATA IN L^2

Consider

$$\mathbf{g} \in L^2(\Gamma)^n, \qquad \int_\Gamma \mathbf{g} \cdot \mathbf{n} = 0$$

Well posedness of this kind of problems has been studied by classic techniques based on integral operators.

Particularly, the Stokes equations were analyzed by Fabes, Kenig and Verchota (1988)

In 2d it was also studied by Hamouda, Temam and Zhang (2017) using variational techniques and “very weak” solutions.

BONDARY DATA IN L^2

Fabes et al. proved that for a Lipschitz domain, given $\mathbf{g} \in L^2(\Gamma)^n$ there exists a unique solution

$$(\mathbf{u}, p) \in L^2(\Omega)^n \times H^{-1}(\Omega)/\mathbb{R}$$

such that

$$\|\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{H^{-1}(\Omega)/\mathbb{R}} \leq C\|\mathbf{g}\|_{L^2(\Gamma)}$$

They also proved

$$(\mathbf{u}, p) \in H^{\frac{1}{2}}(\Omega)^n \times H^{-\frac{1}{2}}(\Omega)/\mathbb{R}$$

The boundary values are attained in the sense of non-tangential convergence a. e. on Γ.

FINITE ELEMENT APPROXIMATION

As we already mentioned, the standard FEM cannot be applied directly. Therefore, we will use a two step procedure:

STEP 1: Aproximate $\mathbf{g}$ by some regularization

STEP 2: Solve the regularized problem by some stable standard FEM.

For Laplace equation this method was analyzed by Apel, Nicaise, Pfefferer.

STEP 1: REGULARIZATION

$h > 0$ is a parameter (later corresponding to the meshes).

Let $\mathbf{g}_h \in H^{\frac{1}{2}}(\Gamma)$, $h \to 0$, be such that

$$\|\mathbf{g} - \mathbf{g}_h\|_{L^2(\Gamma)} \to 0 \quad , \quad \int_\Gamma \mathbf{g}_h \cdot \mathbf{n} = 0$$

Then, for each h we introduce the regularized problem,

$$\begin{aligned} -\Delta \mathbf{u}(h) + \nabla p(h) &= \mathbf{0} \qquad \text{en } \Omega \\ \operatorname{div} \mathbf{u}(h) &= 0 \qquad \text{en } \Omega \\ \mathbf{u}(h) &= \mathbf{g}_h \qquad \text{en } \Gamma \end{aligned}$$

We know that there exists a unique solution

$$(\mathbf{u}(h), p(h)) \in H^1(\Omega)^n \times L^2_0(\Omega)$$

STEP 2: FINITE ELEMENT APPROXIMATION

Consider

$$(\mathbf{U}_h, Q_h) \subset H^1(\Omega)^n \times L^2_0(\Omega) \quad \mathbf{V}_h = \mathbf{U}_h \cap H^1_0(\Omega)^n$$

a stable pair for Stokes. That is, such that the inf-sup condition is satisfied:

$$\inf_{0 \neq p \in Q_h} \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V_h}} \frac{(p, \operatorname{div} \mathbf{v})}{\|\mathbf{v}\|_{H^1(\Omega)} \|p\|_{L^2(\Omega)}} \geq \alpha > 0$$

Classic examples are:

Arnold-Brezzi-Fortin (Mini element): $\mathcal{P}_1 + \mathit{Bubble} - \mathcal{P}_1$

Hood-Taylor: $\mathcal{P}_2 - \mathcal{P}_1$.

STEP 2: FINITE ELEMENT APPROXIMATION

Assume that $\mathbf{g}_h$ is the restriction to Γ of a function in $\mathbf{U}_h$

Then, the approximate solution is given by

Find $(\mathbf{u}_h, p_h) \in \mathbf{U}_h \times Q_h$ con $\mathbf{u}_h|_\Gamma = \mathbf{g}_h$ such that

$$\begin{aligned} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (\operatorname{div} \mathbf{v}_h, p_h) &= \mathbf{0} \qquad \forall \mathbf{v}_h \in \mathbf{U}_h \cap H_0^1(\Omega)^n \\ (\operatorname{div} \mathbf{u}_h, q_h) &= 0 \qquad \forall q_h \in Q_h \end{aligned}$$

Our goal is to analyze the error

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \qquad \text{y} \qquad \|p - p_h\|_{H^{-1}(\Omega)/\mathbb{R}}$$

when $\mathbf{g}_h \longrightarrow \mathbf{g}$

FINITE ELEMENT APPROXIMATION

The error analysis is divided in two parts:

1. Error due to regularization:

$$\|\mathbf{u} - \mathbf{u}(h)\|_{L^2(\Omega)} \qquad \text{y} \qquad \|p - p(h)\|_{H^{-1}(\Omega)/\mathbb{R}}$$

2. FE approximation error:

$$\|\mathbf{u}(h) - \mathbf{u}_h\|_{L^2(\Omega)} \qquad \text{y} \qquad \|p(h) - p_h\|_{H^{-1}(\Omega)/\mathbb{R}}$$

Given a sequence $\mathbf{g}_h$ such that

$$\mathbf{g}_h \in H^{\frac{1}{2}}(\Gamma)^n, \quad \int_\Gamma \mathbf{g}_h \cdot \mathbf{n} = 0, \qquad \|\mathbf{g}_h - \mathbf{g}\|_{0,\Gamma} \to 0$$

We want to estimate the error between the solutions of the original problem and the regularized one:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= 0 \quad \text{en } \Omega \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{en } \Omega \\ \mathbf{u} &= \mathbf{g} \quad \text{en } \Gamma \end{aligned}$$

and

$$\begin{aligned} -\Delta \mathbf{u}(h) + \nabla p(h) &= 0 \quad \text{en } \Omega \\ \operatorname{div} \mathbf{u}(h) &= 0 \quad \text{en } \Omega \\ \mathbf{u}(h) &= \mathbf{g}_h \quad \text{en } \Gamma \end{aligned}$$

THEOREM: If Ω is a convex polygonal or polyhedral domain we have, for $0 \le s < \frac{1}{2}$,

$$\|\mathbf{u} - \mathbf{u}(h)\|_{0,\Omega} + \|p - p(h)\|_{H^{-1}(\Omega)/\mathbb{R}} \le \frac{C}{1-2s} \|\mathbf{g} - \mathbf{g}_h\|_{-s,\Gamma},$$

with a constant C independent of s.
Idea: We introduce the dual problem

$$\begin{aligned} -\Delta\Phi + \nabla q &= \mathbf{u} - \mathbf{u}(h) && \text{en } \Omega \\ \operatorname{div}\Phi &= 0 && \text{en } \Omega \\ \Phi &= 0 && \text{en } \Gamma \end{aligned}$$

It is known that

$$\|\Phi\|_{H^2(\Omega)} + \|q\|_{H^1(\Omega)} \le C\|\mathbf{u} - \mathbf{u}(h)\|_{L^2(\Omega)}$$

$$\|\mathbf{u}-\mathbf{u}(h)\|^2_{L^2(\Omega)} = (\mathbf{u}-\mathbf{u}(h), -\Delta\Phi + \nabla q)$$
$$= (\mathbf{g}-\mathbf{g}_h, \partial_{\mathbf{n}}\Phi)_\Gamma + ((\mathbf{g}-\mathbf{g}_h)\cdot\mathbf{n}, q)_\Gamma$$

$\Gamma = \cup_{i=1}^N \Gamma_i$ Γ_i sides or faces of Γ.

Both terms can be treated in the same way. For example:

$$(\mathbf{g}-\mathbf{g}_h, q\mathbf{n})_\Gamma = \sum_i (\mathbf{g}-\mathbf{g}_h, q\mathbf{n})_{\Gamma_i} \le \sum_i \|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma_i} \|q\mathbf{n}\|_{s,\Gamma_i}$$
$$\le C\left(\sum_i \|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma_i}\right)\|q\|_{1,\Omega} \le \frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma}\|q\|_{1,\Omega}$$

We will prove below the last inequality for $0 \le s < \frac{1}{2}$.

Therefore, we obtain

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}(h)\|^2_{L^2(\Omega)} &= (\mathbf{g}-\mathbf{g}_h,\partial_{\mathbf{n}}\Phi)_\Gamma + ((\mathbf{g}-\mathbf{g}_h)\cdot\mathbf{n},q)_\Gamma \\ &\le \frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma}\left(\|\Phi\|_{2,\Omega}+\|q\|_{H^1(\Omega)}\right) \\ &\le \frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma}\|\mathbf{u}-\mathbf{u}(h)\|_{L^2(\Omega)}\end{aligned}$$

where we have used the a priori estimate for Φ and q.

Given a function $\varphi \in L^2(\Gamma_i)$ we call $\widetilde{\varphi}$ its extension by zero.

$$\widetilde{\varphi} = \begin{cases} \varphi & \text{en} \quad \Gamma_i \\ 0 & \text{en} \quad \Gamma \setminus \Gamma_i \end{cases}$$

Obviously we have $\|\widetilde{\varphi}\|_{0,\Gamma} = \|\varphi\|_{0,\Gamma_i}$

But it is not true that, for $s > 0$, $\|\widetilde{\varphi}\|_{s,\Gamma} = \|\varphi\|_{s,\Gamma_i}$

The fractional norm is not local!

Moreover, if $s \geq 1/2$,

$$\varphi \in H^s(\Gamma_i) \quad \not\Rightarrow \quad \tilde{\varphi} \in H^s(\Gamma)$$

Lemma: If $0 \leq s < 1/2$,

$$\|\tilde{\varphi}\|_{s,\Gamma} \leq \frac{C}{1-2s}\|\varphi\|_{s,\Gamma_i}$$

with C independent of s.

Proof: It is not difficult to see that, for $x \in \Gamma_i$,

$$\int_{\Gamma \setminus \Gamma_i} \frac{1}{|x-y|^{n-1+2s}} dy \leq \frac{C}{d(x,\Gamma_i^c)^{2s}}$$

where $d(x,\Gamma_i^c)$ is the distance from x to $\Gamma_i^c := \Gamma \setminus \Gamma_i$.

Then,

$$|\widetilde{\varphi}|^2_{s,\Gamma} = \int_\Gamma \int_\Gamma \frac{|\widetilde{\varphi}(x) - \widetilde{\varphi}(y)|^2}{|x-y|^{n-1+2s}} dydx$$
$$= |\varphi|^2_{s,\Gamma_i} + 2 \int_{\Gamma_i} \int_{\Gamma \setminus \Gamma_i} \frac{|\varphi(x)|^2}{|x-y|^{n-1+2s}} dydx$$
$$\leq |\varphi|^2_{s,\Gamma_i} + C \int_{\Gamma_i} \frac{|\varphi(x)|^2}{d(x,\Gamma_i^c)^{2s}} dx \leq \frac{C}{(1-2s)^2} \|\varphi\|^2_{s,\Gamma_i}$$

where we have used the Hardy inequality, for $s < 1/2$,

$$\int_{\Gamma_i} \frac{|\varphi(x)|^2}{d(x,\Gamma_i^c)^{2s}} dx \leq \frac{C}{(1-2s)^2} \|\varphi\|^2_{s,\Gamma_i}$$

The dependence in s is optimal.

Consequently, by duality we have,

$$\|f\|_{-s,\Gamma_i} \leq \frac{C}{1-2s}\|f\|_{-s,\Gamma}$$

Indeed,

$$\begin{aligned}
\|f\|_{-s,\Gamma_i} &= \sup_{0\neq\varphi\in H^s(\Gamma_i)} \frac{\int_{\Gamma_i} f\varphi}{\|\varphi\|_{s,\Gamma_i}} \\
&= \sup_{0\neq\varphi\in H^s(\Gamma_i)} \frac{\int_{\Gamma_i} f\widetilde{\varphi}}{\|\widetilde{\varphi}\|_{s,\Gamma}} \frac{\|\widetilde{\varphi}\|_{s,\Gamma}}{\|\varphi\|_{s,\Gamma_i}} \\
&\leq \frac{C}{1-2s}\|f\|_{-s,\Gamma}
\end{aligned}$$

PART 2: FE APPROXIMATION ERROR

THEOREM: If Ω is a convex polygonal or polyhedral domain:

$$\|\mathbf{u}(h) - \mathbf{u}_h\|_{0,\Omega} + \|p(h) - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \leq Ch\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma}.$$

- By standard stability results we have

$$\|\mathbf{u}(h) - \mathbf{u}_h\|_{H^1(\Omega)} + \|p(h) - p_h\|_{L^2(\Omega)} \leq C\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma}$$

- Then we use duality: the key point is to use that

$$\forall f \in L^2_0(\Omega) \cap H^1_0(\Omega) \qquad \exists \mathbf{v} \in H^2_0(\Omega)^n$$

such that,

$$\operatorname{div} \mathbf{v} = f \qquad , \qquad \|\mathbf{v}\|_{H^2_0(\Omega)} \leq C\|f\|_{H^1(\Omega)}$$

which can be proved using Bogovskii's solution.

TOTAL ERROR

Summing up:

THEOREM: For $0 \leq s < 1/2$,

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}+\|p-p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \leq \frac{C}{1-2s}\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma}+Ch\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma}.$$

For example, if we take $\mathbf{g}_h$ as the L^2 projection of $\mathbf{g}$ over the continuous piece-wise linear functions we have (assuming quasi-uniform meshes):

$$\|\mathbf{g}-\mathbf{g}_h\|_{-s,\Gamma} \leq Ch^{s+t}\|\mathbf{g}\|_{t,\Gamma}, \qquad s,t \in [0,1]$$

$$\|\mathbf{g}_h\|_{t,\Gamma} \leq C\|\mathbf{g}\|_{t,\Gamma}, \qquad t \in [0,1]$$

$\mathbf{g}_h$: L^2 PROJECTION

For $0 \le t \le 1/2$ (inverse inequalities):

$$\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma} \le Ch^{t-\frac{1}{2}}\|\mathbf{g}_h\|_{t,\Gamma} \le Ch^{t-\frac{1}{2}}\|\mathbf{g}\|_{t,\Gamma}$$

and then we obtain:

If $0 \le t \le 1/2, \quad 0 \le s < 1/2, \quad \mathbf{g} \in H^t(\Gamma)^n$,

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} + \|p-p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \le \frac{C}{1-2s}h^{s+t}\|\mathbf{g}\|_{t,\Gamma} + Ch^{\frac{1}{2}+t}\|\mathbf{g}\|_{t,\Gamma}$$

$\mathbf{g}_h$: L^2 PROJECTION

Extrapolation: Take $s = \frac{1}{2} + \frac{1}{\log h} < \frac{1}{2}$ $\quad (s \to 1/2 \text{ when } h \to 0)$

THEOREM: If $0 \leq t \leq 1/2$ and $\mathbf{g} \in H^t(\Gamma)^n$,

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|p - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \leq C|\log h| h^{\frac{1}{2}+t} \|\mathbf{g}\|_{t,\Gamma}$$

In particular, taking $t = 0$:

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|p - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \leq C|\log h| h^{\frac{1}{2}} \|\mathbf{g}\|_{0,\Gamma}$$

which is almost optimal since $\mathbf{u} \in H^{\frac{1}{2}}(\Omega)$, and it is not better in general.

Of course, the drawback of the L^2 projection is that it is not local.

To compute it we have to solve a global problem.

$\mathbf{g}_h$ CARSTENSEN INTERPOLATION

A better option is to use the following approximation introduced by Carstensen:

$$\mathbf{g} \qquad \longrightarrow \qquad \mathbf{g}_h = \sum_{\text{nodes } z \text{ in } \Gamma} \frac{(\mathbf{g}, \varphi_z)}{(1, \varphi_z)} \varphi_z$$

where $\{\varphi_z\}$ is the nodal basis associated with the mesh on the boundary.

- The computation is local.
- It satisfies the same error estimates than the L^2 projection and therefore we have the same estimates for the total error.

$\mathbf{g}_h$ LAGRANGE INTERPOLATION

In practice it is natural to use the point values of **g** where they are well defined.

For example, if **g** is piecewise smooth (as in the Lid-driven cavity), it is natural to use the Lagrange interpolation, with appropriate modification at jumps (taking some average value for example). It is possible to choose those values in such a way that $\mathbf{g}_h$ satisfy the compatibility conditiion:

$$\int_\Gamma \mathbf{g}_h \cdot \mathbf{n} = 0$$

$\mathbf{g}_h$ LAGRANGE INTERPOLATION

The error analysis is more complicated and the error estimates are weaker than those valid for the L^2 projection. We can prove:

THEOREM:

$$s < \frac{1}{2}, n = 2 : \quad \|\mathbf{g} - \mathbf{g}_h\|_{-s,\Gamma} \leq \frac{C}{\sqrt{1-2s}} h^{\frac{1}{2}+s} \sum_{i=1}^{N} \|\mathbf{g}\|_{H^1(\Gamma_i)}$$

$$s < \frac{1}{2}, n = 3 : \quad \|\mathbf{g} - \mathbf{g}_h\|_{-s,\Gamma} \leq \frac{C}{1-2s} h^{\frac{1}{2}+s} \sum_{i=1}^{N} \|\mathbf{g}\|_{H^2(\Gamma_i)}$$

$$\|\mathbf{g}_h\|_{\frac{1}{2},\Gamma} \leq C|\log h_{min}| \sum_{i=1}^{N} \|\mathbf{g}\|_{H^1(\Gamma_i)}$$

$\mathbf{g}_h$ LAGRANGE INTERPOLATION

Choosing $s = \frac{1}{2} + \frac{1}{\log h} < \frac{1}{2}$ $\quad (s \to 1/2$ when $h \to 0)$:

THEOREM:

- In the $2d$ case, if $\mathbf{g} \in H^1(\Gamma_i)^2$ for all i,

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|p - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \leq Ch\,|\log h|^{3/2} \sum_{i=1}^{N} \|\mathbf{g}\|_{H^1(\Gamma_i)}$$

- In the $3d$ case, if $\mathbf{g} \in H^2(\Gamma_i)^3$ for all i,

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|p - p_h\|_{H^{-1}(\Omega)/\mathbb{R}} \leq Ch\,|\log h|^2 \sum_{i=1}^{N} \|\mathbf{g}\|_{H^2(\Gamma_i)}$$

IDEA OF THE PROOF IN 2d

We use the embedding theorem:

$$s < \frac{1}{2} \quad , \quad q = \frac{2}{1-2s} \quad \Longrightarrow \quad H^s(\Gamma) \subset L^q(\Gamma)$$

and

$$\|\phi\|_{L^q(\Gamma)} \leq \frac{C}{\sqrt{1-2s}} \|\phi\|_{s,\Gamma}, \qquad \forall \phi \in H^s(\Gamma)$$

Take $p = \frac{2}{1+2s}$ and $q = \frac{2}{1-2s}$ its dual exponent, then

$$\begin{aligned} \|\mathbf{g} - \mathbf{g}_h\|_{-s,\Gamma} &= \sup_{\phi:\|\phi\|_{s,\Gamma}=1} \int_\Gamma (\mathbf{g} - \mathbf{g}_h)\,\phi \\ &\leq \sup_{\phi:\|\phi\|_{s,\Gamma}=1} \|\mathbf{g} - \mathbf{g}_h\|_{L^p(\Gamma)} \|\phi\|_{L^q(\Gamma)} \\ &\leq \frac{C}{\sqrt{1-2s}} \|\mathbf{g} - \mathbf{g}_h\|_{L^p(\Gamma)}. \end{aligned}$$

IDEA OF THE PROOF IN $2d$

Let Γ_S be the union of elements containing the singular points:

$$\|\mathbf{g} - \mathbf{g}_h\|_{L^p(\Gamma_S)} \leq \|\mathbf{g} - \mathbf{g}_h\|_\infty |\Gamma_S|^{\frac{1}{p}}$$

$$\|\mathbf{g} - \mathbf{g}_h\|_\infty \leq C\|\mathbf{g}\|_\infty \leq C\sum_{i=1}^{N} \|\mathbf{g}\|_{H^1(\Gamma_i)} \quad , \quad |\Gamma_S| \sim h$$

Away from the singularities we use standard interpolation error estimates. Then,

$$\|\mathbf{g} - \mathbf{g}_h\|_{-s,\Gamma} \leq \frac{C}{\sqrt{1-2s}} h^{\frac{1}{p}} \sum_{i=1}^{N} \|\mathbf{g}\|_{H^1(\Gamma_i)}$$

and taking $s \to \frac{1}{2}$, $p = \frac{2}{1+2s} \to 1$ we conclude the proof.

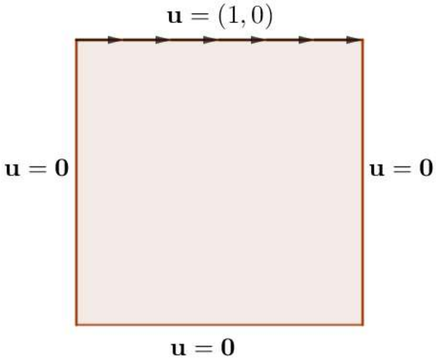
IDEA OF THE PROOF IN 3d

In 3d an analogous argument gives a suboptimal order due to the fact that the exponent q in the embedding theorem depends on the dimension.

SOLUTION: Use a Hardy type inequality instead of an embedding theorem. (We thank Pablo De Nápoli who suggested us this argument).

This argument can also be applied in 2d but it gives a suboptimal result in terms of s (or in terms of the power of $|\log h|$ after taking $s \to \frac{1}{2}$)

We present some numerical experiments for the problem



NUMERICAL RESULTS

We approximate **g** by Lagrange interpolation (averaging at the jumps) and we use two classic methods (known to be stable).

Mini-Element ($\mathcal{P}_1 + B - \mathcal{P}_1$) y Hood-Taylor ($\mathcal{P}_2 - \mathcal{P}_1$)

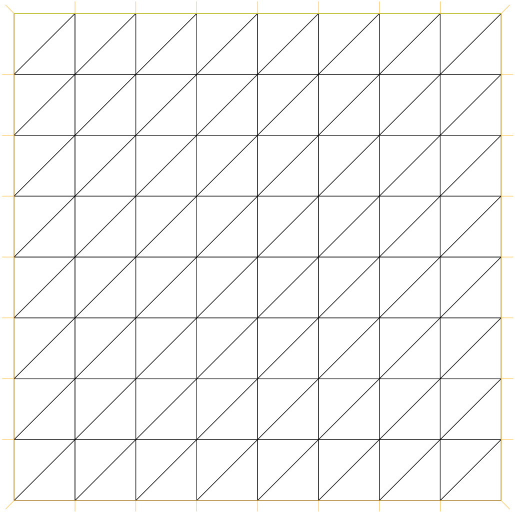
Our error estimates predict, for quasi-uniform mehes, an error $O(h)$ up to a logarithmic factor.

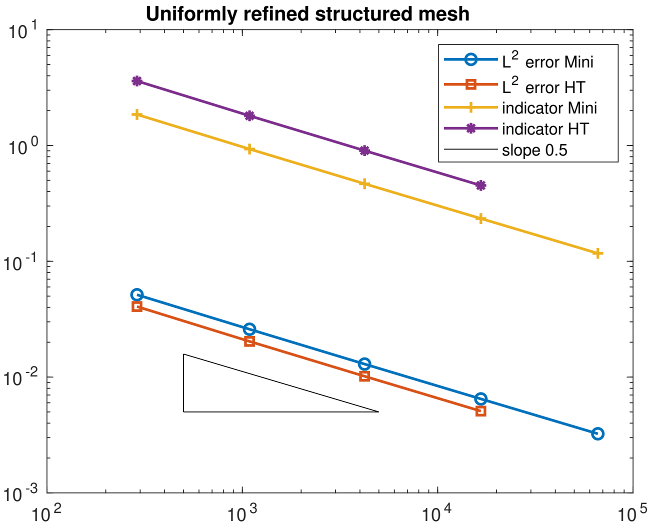
It is known that $\mathbf{u} \in H^s(\Omega)^2$ for any $s < 1$ but $\mathbf{u} \notin H^1(\Omega)^2$. Therefore, $O(h)$ is the best possible.

We have also introduced and analyzed a posteriori error estimates.

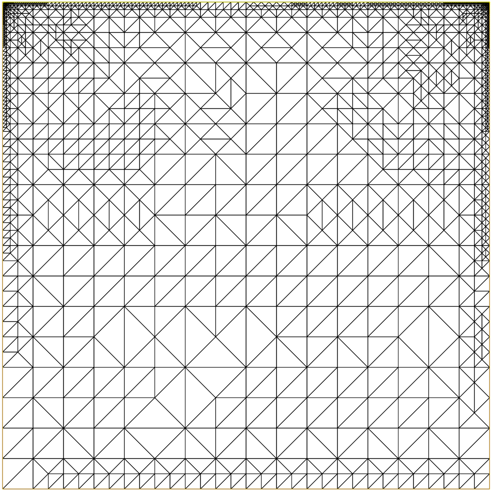
We present also some numerical results obtained by an adaptive procedure based on our estimators.

UNIFORM MESHES

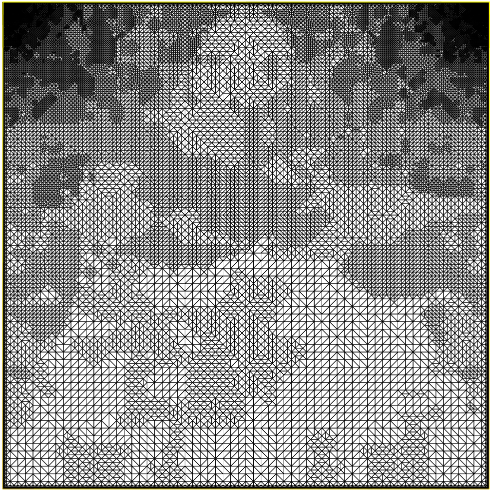



Uniformly refined structured mesh
L2 error Mini
L2 error HT
indicator Mini
indicator HT
slope 0.5
10 1
10 0
10 -1
10 -2
10 -3
10 2
10 3
10 4
10 5

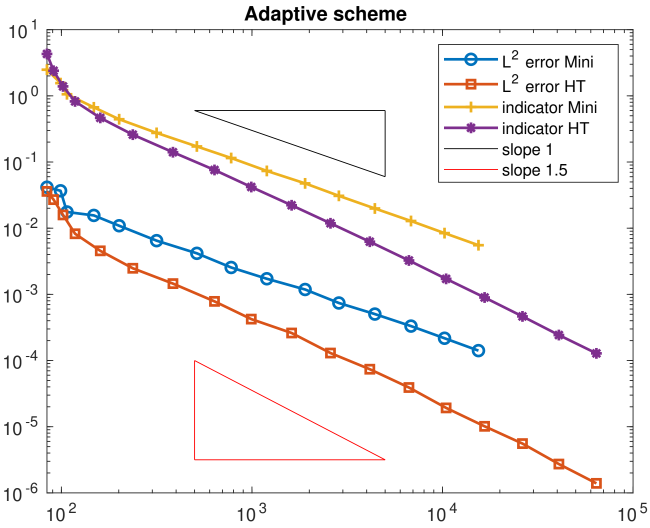
ADAPTIVE MESHES



ADAPTIVE MESHES



ORDER OF THE VELOCITY L^2 ERROR



ORDER OF THE VELOCITY L^2 ERROR

- UNIFORM MESHES:

  Mesh size h, number of nodes $N = h^{-2}$

  Mini element: $O(h) = O(N^{-1/2})$
  Hood-Taylor: $O(h) = O(N^{-1/2})$

  It is optimal: $\mathbf{u} \in H^{1-\varepsilon} \setminus H^1$

- ADAPTIVE MESHES:

  Mini element: $O(N^{-1})$
  Hood- Taylor: $O(N^{-3/2})$

  It is optimal: same order with respect to N as that for a smooth solution in uniform meshes, which is $O(h^3)$ for HT and $O(h^2)$ for Mini.

OPEN PROBLEMS AND FURTHER RESEARCH

- Non convex domains: For the Laplace equation a priori estimates has been obtained by Apel, Nicaise and Pfefferer. Probably similar results can be proved for Stokes.

- To obtain efficient and reliable a posteriori estimators for non-convvex domains seems to be a difficult problem: as far as I know this has not been done for Laplace equation because of the duality arguments needed to prove L^2-estimates and the lack of regularity of the dual problem.

- Other possible analysis: Use weighted Sobolev spaces. For examplle, in the Lid-driven cavity problem, $\mathbf{u} \notin H^1$ but it belongs to weighted H^1 with weight $|x|^\alpha$.

THANK YOU VERY MUCH!