TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

Gunar Matthies
Fakultät Mathematik, Institut für Numerische Mathematik

Local Projection Stabilisation

Irish Numerical Analysis Forum
25 March 2021

Contents

First works on local projection stabilisation

Becker, Braack, 2001
A finite element pressure gradient stabilization for the Stokes equations based on local projections

Becker, Braack, 2004
A two-level stabilization scheme for the Navier–Stokes equations

Braack, Burman, 2006
Local projection stabilization for the Oseen problem and its interpretation as a variational multiscale method

Convection-diffusion equations and weak formulation

convection-diffusion equation

$$-\varepsilon\triangle u + b\cdot\nabla u + cu = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

bilinear form

$$a(u,v) := \varepsilon(\nabla u,\nabla v) + (b\cdot\nabla u, v) + (cu, v)$$

weak formulation

Find $u \in V := H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \qquad \forall v\in V$$

unique solvability provided

$$c - \frac{1}{2}\,\mathrm{div}\, b \geq c_0 \geq 0$$

Discrete problem

$\{\mathcal{T}_h\}$: family of shape-regular triangulations of domain Ω

conforming finite element space $V_h \subset V$ on $\mathcal{T}_h$ of order r in H^1-norm

discrete problem (without stabilisation)

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

observation: unphyscial oscillations unless h is very small

idea: add stabilising term

Local projection stabilisation

$\{\mathcal{M}_h\}$: family of shape-regular and non-overlapping macro decompositions of Ω

on each macro $M \in \mathcal{M}_h$

- finite dimensional space $D(M)$
- local L^2 projection $\pi_M : L^2(M) \to D(M)$
- fluctuation operator $\kappa_M w := w - \pi_M w$

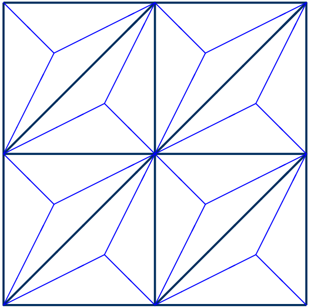
approximation property of κ_M

$$\|\kappa_M q\|_{0,M} \le C\, h_M^\ell |q|_{\ell,M}, \qquad q \in H^\ell(M),\ 0 \le \ell \le r$$

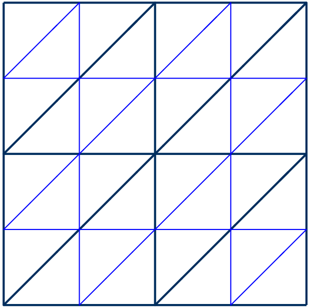
ensured for $P_{r-1}(M) \subset D(M)$

Choice of $\mathcal{M}_h$

two-level approach: $\mathcal{T}_h$ is refinement of $\mathcal{M}_h$



barycentric refinement



regular (red) refinement

one-level approach: $\mathcal{M}_h = \mathcal{T}_h$

Stabilised discrete problem

stabilisation term

$$s_h(u_h, v_h) := \sum_{M \in \mathcal{M}_h} \delta_M \big(\kappa_M(b \cdot \nabla u_h), \kappa_M(b \cdot \nabla v_h)\big)_M$$

with non-negative constants δ_M, $M \in \mathcal{M}_h$, to be fixed later

stabilised bilinear form

$$a_h(u, v) := a(u, v) + s_h(u, v), \qquad u, v \in V$$

stabilised discrete problem

Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

choice $\delta_M = 0$ for all $M \in \mathcal{M}_h$: standard Galerkin discretisation

Solvability and consistency

assumption

$$c - \frac{1}{2}\operatorname{div} b \geq c_0 > 0$$

norm on V

$$|\!|\!| v |\!|\!| := \left(\varepsilon |v|_1^2 + c_0 \|v\|_0^2 + s_h(v,v)\right)^{1/2} = \left(\varepsilon |v|_1^2 + c_0 \|v\|_0^2 + \sum_{M \in \mathcal{M}_h} \delta_M \|\kappa_M(b \cdot \nabla v)\|_{0,M}^2\right)^{1/2}$$

coercivity

$$a_h(v,v) \geq |\!|\!| v |\!|\!|^2, \qquad v \in V$$

unique discrete solution $u_h \in V_h$

consistency error

$$a_h(u - u_h, v_h) = s_h(u, v_h)$$

Error analysis

interpolation operator $i_h : V \to V_h$ with usual error estimates

$$|q - i_h q|_{\ell,M} \le C h_M^{r+1-\ell} \|q\|_{r+1,M}, \qquad M \in \mathcal{M}_h,\ q \in H^{r+1}(M),\ \ell = 0, 1,$$

triangle inequality

$$\|u - u_h\| \le \|u - i_h u\| + \|i_h u - u_h\|$$

handling of discrete error $w_h := i_h u - u_h$

$$\begin{aligned} \|w_h\|^2 = \|i_h u - u_h\|^2 &\le a_h(i_h u - u_h, i_h u - u_h) \\ &= a_h(i_h u - u, w_h) + a_h(u - u_h, w_h) \\ &= a_h(i_h u - u, w_h) + s_h(u, w_h) \end{aligned}$$

estimate term by term

Interpolation error

standard techniques provide

$$\|u - i_h u\|^2 = \varepsilon |u - i_h u|_1^2 + c_0 \|u - i_h u\|_0^2 + \sum_{M \in \mathcal{M}_h} \delta_M \|\kappa_M (b \cdot \nabla (u - i_h u))\|_{0,M}^2$$

$$\leq C \sum_{M \in \mathcal{M}_h} \left(\varepsilon + c_0 h_M^2 + \delta_M b_M^2\right) h_M^{2r} \|u\|_{r+1,M}^2$$

using L^2 stability of κ_M

interesting case $\varepsilon \leq C h_M$: condition $\delta_M \leq C h_M$ for all M ensures

$$\|u - i_h u\| \leq C \left(\sum_{M \in \mathcal{M}_h} h_M^{2r+1} \|u\|_{r+1,M}^2 \right)^{1/2} \leq C h^{r+1/2} \|u\|_{r+1}$$

Estimate of symmetric terms

diffusion term

$$\left|\varepsilon\big(\nabla(i_h u - u_h), \nabla w_h\big)\right| \le C \left(\sum_{M\in\mathcal{M}_h} \varepsilon h_M^{2r} \|u\|_{r+1,M}^2 \right)^{1/2} \|w_h\|$$

reaction term

$$\left|\big(c(i_h u - u), w_h\big)\right| \le C \left(\sum_{M\in\mathcal{M}_h} c_M^2 h_M^{2r+2} \|u\|_{r+1,M}^2 \right)^{1/2} \|w_h\|$$

stabilisation term

$$s_h(i_h u - u, w_h) = \sum_{M\in\mathcal{M}_h} \delta_M \big(\kappa_M (b\cdot\nabla(i_h u - u)), \kappa_M (b\cdot\nabla w_h)\big)_M$$

$$\le C \left(\sum_{M\in\mathcal{M}_h} \delta_M b_M^2 h_M^{2r} \|u\|_{r+1,M}^2 \right)^{1/2} \|w_h\|$$

Estimate of consistency error

using approximation property of κ_M

$$
\begin{aligned}
s_h(u, w_h) &= \sum_{M \in \mathcal{M}_h} \delta_M \big(\kappa_M(b \cdot \nabla u), \kappa_M(b \cdot \nabla w_h)\big)_M \\
&\le \left(\sum_{M \in \mathcal{M}_h} \delta_M \left\| \kappa_M(b \cdot \nabla u) \right\|_{0,M}^2 \right)^{1/2} \left(\sum_{M \in \mathcal{M}_h} \delta_M \left\| \kappa_M(b \cdot \nabla w_h) \right\|_{0,M}^2 \right)^{1/2} \\
&\le C \left(\sum_{M \in \mathcal{M}_h} \delta_M h_M^{2r} \| b \cdot \nabla u \|_{r,M}^2 \right)^{1/2} \| w_h \| \\
&\le C \left(\sum_{M \in \mathcal{M}_h} \delta_M \widetilde{b}_M^2 h_M^{2r} \| u \|_{r+1,M}^2 \right)^{1/2} \| w_h \|
\end{aligned}
$$

Estimate of convective term

without integration by parts

$$\big|(b\cdot\nabla(i_hu-u),w_h)\big| \le \|b\|_\infty |i_hu-u|_1\|w_h\|_0 \le Ch^r\|w_h\|_0 \le Ch^r|||w_h|||$$

integration by parts

$$(b\cdot\nabla(i_hu-u),w_h) = -(b\cdot\nabla w_h, i_hu-u) - ((i_hu-u)\operatorname{div} b, w_h)$$

last term

$$\big|-(i_hu-u)\operatorname{div} b,w_h)\big| \le \|\operatorname{div} b\|_\infty\|i_hu-u\|_0\|w_h\|_0 \le Ch^{r+1}\|w_h\|_0 \le Ch^{r+1}|||w_h|||$$

first term

$$\big|-(b\cdot\nabla w_h, i_hu-u)\big| \le \|b\|_\infty|w_h|_1\|i_hu-u\|_0 \le \begin{cases} Ch^{-1}\|w_h\|_0\|i_hu-u\|_0 & \le Ch^r|||w_h||| \\ C\varepsilon^{1/2}|w_h|_1\varepsilon^{-1/2}\|i_hu-u\|_0 & \le C\dfrac{h^{r+1}}{\varepsilon^{1/2}}|||w_h||| \end{cases}$$

non-optimal estimate

Key in analysis

assume existence of special interpolation operator j_h with

- usual approximation properties

$$|w - j_h w|_{\ell,M} \le C h_M^{r+1-\ell} \|w\|_{r+1,M}, \quad M \in \mathcal{M}_h,\, w \in H^{r+1}(M),\, \ell = 0, 1,$$

- additional orthogonality

$$(w - j_h w, q_h)_M = 0, \qquad q_h \in D(M),\, w \in H^{r+1}(M)$$

conditions ensuring existence of j_h will be discussed soon

Improved estimate of convective term

rewriting first term after integration by parts

$$
\begin{aligned}
(b\cdot\nabla w_h, j_h u - u) &= \sum_{M\in\mathcal{M}_h} (b\cdot\nabla w_h, j_h u - u)_M \\
&= \sum_{M\in\mathcal{M}_h} \big(b\cdot\nabla w_h - \pi_M(b\cdot\nabla w_h), j_h u - u\big)_M \\
&= \sum_{M\in\mathcal{M}_h} \big(\kappa_M(b\cdot\nabla w_h), j_h u - u\big)_M \\
&\le \left(\sum_{M\in\mathcal{M}_h} \delta_M \left\|\kappa(b\cdot\nabla w_h)\right\|_{0,M}^2\right)^{1/2} \left(\sum_{M\in\mathcal{M}_h} \delta_M^{-1} \|j_h u - u\|_{0,M}^2\right)^{1/2} \\
&\le C \left(\sum_{M\in\mathcal{M}_h} \delta_M^{-1} h_M^{2r+2} \|u\|_{r+1,M}^2\right)^{1/2} \| w_h \|
\end{aligned}
$$

Choice of stabilisation parameters

collecting all estimates

$$\|u - u_h\| \le C \left(\sum_{M \in \mathcal{M}_h} \left(\varepsilon + (c_0 + c_M^2) h_M^2 + \delta_M (b_M^2 + \widetilde{b}_M^2) + \delta_M^{-1} h_M^2 \right) h_M^{2r} \|u\|_{r+1,M}^2 \right)^{1/2}$$

optimal choice for δ_M

$$\delta_M \sim h_M$$

final estimate

$$\|u - u_h\| \le C \left(\sum_{M \in \mathcal{M}_h} (\varepsilon + h_M) h_M^{2r} \|u\|_{r+1,M}^2 \right)^{1/2}$$

in interesting case $\varepsilon \le Ch$

$$\|u - u_h\| \le C h^{r+1/2} \|u\|_{r+1}$$

On existence of special interpolation operator

sufficient conditions for existence of j_h

- interpolation operator i_h with usual approximation properties

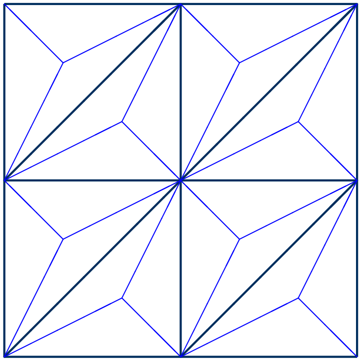
$$|w - i_h w|_{\ell,M} \le C h_M^{r+1-\ell} \|w\|_{r+1,M}, \quad M \in \mathcal{M}_h,\, w \in H^{r+1}(M),\, \ell = 0, 1,$$

- inf-sup condition

$$\inf_{q_h \in D(M)} \sup_{v_h \in Y(M)} \frac{(v_h, q_h)}{\|v_h\|_{0,M} \|q_h\|_{0,M}} \ge \beta^* > 0, \qquad M \in \mathcal{M}_h$$

where $Y(M) := Y_h|_M \cap H_0^1(M)$ is the local bubble part of Y_h on M

Two-level approach on barycentrically refined simplices



spaces

- $Y_h := \{v \in H^1(\Omega) \,:\, v|_K \in P_r(K),\, K \in \mathcal{T}_h\}$
- $D(M) := P_{r-1}(M)$

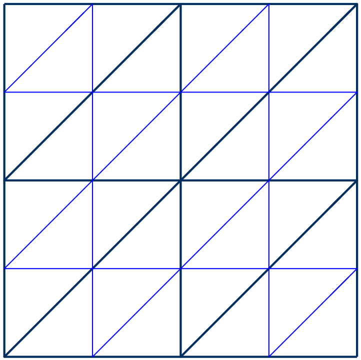
fulfil inf-sup condition

constructive proof of inf-sup condition

M., Skrzypacz, Tobiska, 2007

works also for tetrahedra

Two-level approach on regularly refined triangles



spaces

- $Y_h := \{v \in H^1(\Omega) \,:\, v|_K \in P_r(K),\, K \in \mathcal{T}_h\}$
- $D(M) := P_{r-1}(M)$

fulfil inf-sup condition

technical proof of inf-sup condition

M., Tobiska, 2012

general case for regularly refined tetrahedra still open

One-level approach on quadrilaterals and hexahedra

choice: $\mathcal{M}_h = \mathcal{T}_h$

space on reference cell $\widehat{K} = (-1,1)^d$

$$\widehat{Q}_r^{\text{bubble}} = Q_r(\widehat{K}) \oplus \text{span}(\widehat{b}\, x_i^{r-1},\ i = 1,\ldots,d\}$$

with lowest order bubble function $\widehat{b} \in Q_2(\widehat{K}) \cap H_0^1(\widehat{K})$

mapped spaces

$$Y_h := \left\{ v \in H^1(\Omega)\ :\ v|_K \circ F_K \in \widehat{Q}_r^{\text{bubble}},\ K \in \mathcal{T}_h \right\}$$

$$D(K) := \left\{ q|_K \circ F_K \in P_{r-1}(\widehat{K}) \right\}$$

fulfil inf-sup condition

constructive proof of inf-sup condition

M., Skrzypacz, Tobiska, 2007

Remarks

other choices of stabilising term

$$s_h(u,v) = \sum_{M\in\mathcal{M}_h} \delta_M (\kappa_M \nabla u, \kappa_M \nabla v)_M$$

$$s_h(u,v) = \sum_{M\in\mathcal{M}_h} \delta_M \big(\kappa_M (b_M\cdot\nabla u), \kappa_M (b_M\cdot\nabla v)\big)_M$$

with a macro-wise constant approximation b_M of convection b: Knobloch, 2009

overlapping macros: Knobloch, 2010

local projection stabilisation related to subgrid modelling by Guermond
(fluctuations of gradient here vs. gradient of fluctations there)

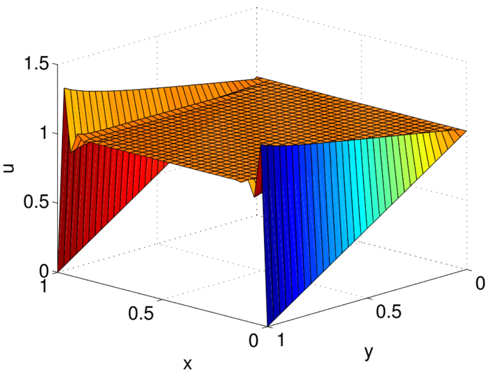
LPS-norm is as strong as SUPG-norm: Knobloch, Tobiska, 2011

Numerical example

problem with mixed boundary conditions

M., Skrzypacz, Tobiska, 2008

$$
\begin{aligned}
-10^{-7}\triangle u + \begin{pmatrix} 0 \\ 1+x^2 \end{pmatrix} \cdot \nabla u &= f && \text{in } (0,1)^2, \\
\frac{\partial u}{\partial n} &= 0 && \text{if } y = 1, \\
u &= 1-y && \text{if } y \neq 1
\end{aligned}
$$

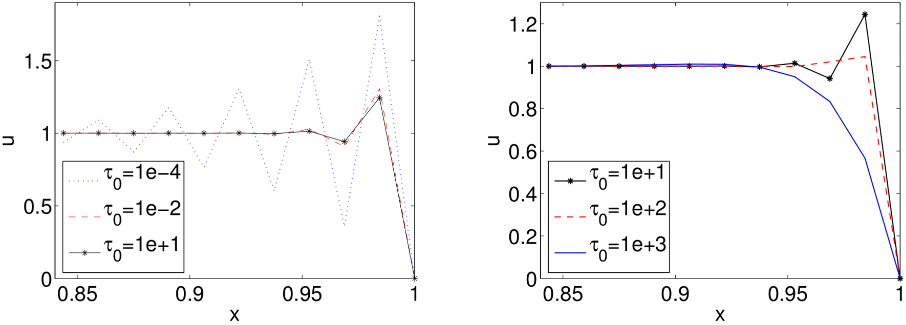


parabolic layers along $x = 0$ and $x = 1$

Numerical results

squares, one-level approach — M., Skrzypacz, Tobiska, 2008

$V_h = Q_1^{\text{bubble}}$, $D(K) = P_0(K)$, full gradient, $\delta_M = \tau_0 h_M$



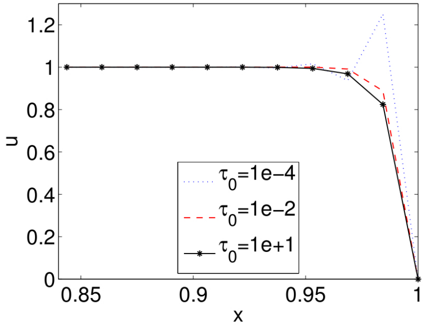
Numerical results

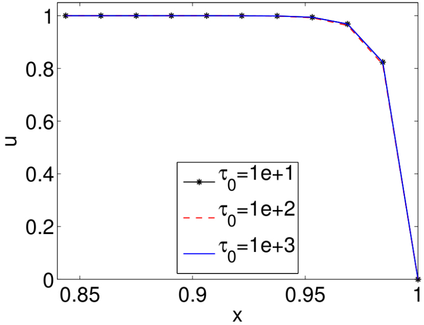
squares, one-level approach

M., Skrzypacz, Tobiska, 2008

$V_h = Q_2^{\text{bubble}}$, $D(K) = P_1(K)$, full gradient, $\delta_M = \tau_0 h_M$

only linear part shown





Oseen equations

Oseen equations with homogeneous Dirichlet boundary condition

$$\begin{aligned} -\nu\triangle u + (b\cdot\nabla)u + \sigma u + \nabla p &= f && \text{in } \Omega,\\ \operatorname{div} u &= 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

physical quantities

- velocity u
- pressure p

assumptions on problem data

- given velocity field b: $b \in W^{1,\infty}(\Omega)$ and $\operatorname{div} b = 0$
- viscosity ν: $0 < \nu$, usually: $\nu \ll 1$
- reaction coefficient σ: $0 \le \sigma$

Weak formulation of Oseen equations

spaces: $V := H_0^1(\Omega)^d$, $Q := L_0^2(\Omega)$

bilinear form

$$A\big((u,p);(v,q)\big) := \nu(\nabla u, \nabla v) + \big((b\cdot\nabla)u, v\big) + \sigma(u,v) - (p, \operatorname{div} v) + (q, \operatorname{div} u)$$

weak formulation

Find $(u,p) \in V \times Q$ such that

$$A\big((u,p);(v,q)\big) = (f,v) \qquad \forall (v,q) \in V \times Q$$

unique solvability due to inf-sup condition for (V,Q)

Discrete problem

$\{\mathcal{T}_h\}$: family of shape-regular triangulations

conforming discrete spaces

- velocity $V_h \subset V$
- pressure $Q_h \subset Q$

discrete problem without any stabilisation

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$A\big((u_h, p_h); (v_h, q_h)\big) = (f, v_h) \qquad \forall (v_h, q_h) \in V_h \times Q_h$$

Drawbacks

- uniform discrete inf-sup condition

$$\exists \beta > 0 \; \forall h \quad \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(\operatorname{div} v_h, q_h)}{\|q_h\|_0 \, |v_h|_1} \geq \beta$$

  might be violated

- dominating convection

- [fundamental invariance property

$$f \to f + \nabla \Phi \qquad \Longrightarrow \qquad (u_h, p_h) \to (u_h, p_h + j_h \Phi)$$

  might be violated]

Stabilised discrete problem: equal-order case

equal-order discrete spaces based on scalar finite element space $Y_h \subset H^1(\Omega)$ of order r:

- velocity $V_h := Y_h^d \cap V$
- pressure $Q_h := Y_h \cap Q$

stabilised discrete problem

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$A\big((u_h, p_h); (v_h, q_h)\big) + S_h\big((u_h, p_h); (v_h, q_h)\big) = (f, v_h) \quad \forall (v_h, q_h) \in V_h \times Q_h$$

separate control on fluctuations of streamline derivative, divergence, and pressure gradient by

$$\begin{aligned} S_h\big((u_h, p_h); (v_h, q_h)\big) := \sum_{M \in \mathcal{M}_h} \Big[& \tau_M \big(\kappa_M((b \cdot \nabla) u_h), \kappa_M (b \cdot \nabla) v_h\big)_M \\ & + \mu_M \big(\kappa_M(\operatorname{div} u_h), \kappa_M(\operatorname{div} v_h)\big)_M \\ & + \alpha_M \big(\kappa_M(\nabla p_h), \kappa_M(\nabla q_h)\big)_M \Big] \end{aligned}$$

Stability

mesh dependent norm

$$|||(v,q)||| := \left(\nu|v|_1^2 + \sigma\|v\|_0^2 + \alpha\|q\|_0^2 + S_h\big((v,q),(v,q)\big)\right)^{1/2}, \qquad \frac{1}{\alpha} := \nu + \sigma C_F^2 + \frac{2b_\infty^2 C_F^2}{\nu + \sigma C_F^2}$$

with Friedrichs' constant C_F

assumption on stabilisation parameters

$$\max_{M\in\mathcal{M}_h}\left(\tau_M, \mu_M, \frac{h_M^2}{\alpha_M}\right) \le C$$

There exists a constant $\beta > 0$ independent of ν and h such that

$$\inf_{(v_h,q_h)\in V_h\times Q_h}\ \sup_{(w_h,r_h)\in V_h\times Q_h} \frac{(A+S_h)((v_h,q_h);(w_h,r_h))}{|||(v_h,q_h)|||\ |||(w_h,r_h)|||} \ge \beta.$$

M., Skrzypacz, Tobiska, 2007

Consistency and error estimate

assumptions

- $\tau_M \sim h_M$, $\mu_M \sim h_M$, $\alpha_M \sim h_M$
- b macro-wise smooth

weak consistency

$$\left|(A+S_h)\big((u-u_h,p-p_h);(w_h,r_h)\big)\right| = \left|S_h\big((u,p);(w_h,r_h)\big)\right| \le C\left(\sum_{M\in\mathcal{M}_h} h_M^{2r+1}\left(\|u\|^2_{r+1,M}+\|p\|^2_{r+1,M}\right)\right)^{1/2} |||(w_h,r_h)|||$$

error estimate

$$|||(u-u_h,p-p_h)||| \le C\left(\nu^{1/2}+h^{1/2}\right)h^r\left(\|u\|_{r+1}+\|p\|_{r+1}\right)$$

M., Skrzypacz, Tobiska, 2007

Stabilised discrete problem: inf-sup stable case

finite element spaces

- velocity $V_h \subset V$: elements of order r
- pressure $Q_h \subset Q$: elements of order $r-1$

fulfilling the uniform discrete inf-sup condition

stabilised discrete problem

Find $(u_h, p_h) \in V_h \times Q_h$ such that

$$A_h\big((u_h, p_h); (v_h, q_h)\big) = (f, v_h) \quad \forall (v_h, q_h) \in V_h \times Q_h$$

with

$$A_h\big((u_h, p_h); (v_h, q_h)\big) := A\big((u_h, p_h); (v_h, q_h)\big) + S_h(u_h, v_h)$$

Stabilisation term

version a: separate control on fluctuations of derivative in streamline direction and divergence

$$S_h^a(u,v) := \sum_{M\in\mathcal{M}_h} \Big(\tau_M(\kappa_M^1(b_M\cdot\nabla)u, \kappa_M^1(b_M\cdot\nabla)v)_M + \mu_M(\kappa_M^2 \,\mathrm{div}\, u, \kappa_M^2 \,\mathrm{div}\, v)_M\Big)$$

with b_M as piecewise constant approximation of b

version b: control on fluctuations of the gradient

$$S_h^b(u,v) := \sum_{M\in\mathcal{M}_h} \gamma_M(\kappa_M^3\nabla u, \kappa_M^3\nabla v)_M$$

on each $M \in \mathcal{M}_h$:

- finite dimensional spaces $D_1(M)$, $D_2(M)$, $D_3(M)$
- local L^2 projections $\pi_M^i : L^2(M) \to D_i(M)$, $i = 1, 2, 3$
- fluctuation operators $\kappa_M^i w := w - \pi_M^i w$, $i = 1, 2, 3$

approximation property of κ_M^i, $i = 1, 2, 3$:

$$\|\kappa_M^i q\|_{0,M} \le C\, h_M^\ell |q|_{\ell,M}, \qquad q \in H^\ell(M),\ 0 \le \ell \le s_i$$

Stability

mesh dependent norm

$$|||(v,q)||| := \left(\nu|v|_1^2 + \sigma\|v\|_0^2 + \alpha\|q\|_0^2 + S_h(v,v)\right)^{1/2}, \qquad \frac{1}{\alpha} := \nu + \sigma C_F^2 + \frac{2b_\infty^2 C_F^2}{\nu + \sigma C_F^2}$$

with Friedrichs' constant C_F

assumption on stabilisation parameters

$$\max_{M\in\mathcal{M}_h} \left(\tau_M \|b\|_{0,\infty,M}^2 + \mu_M d\right) \le \frac{C^a}{\alpha}, \qquad \max_{M\in\mathcal{M}_h} \gamma_M \le \frac{C^b}{\alpha}$$

There exists a constant $\beta > 0$ independent of ν and h such that

$$\inf_{(v_h,q_h)\in V_h\times Q_h} \sup_{(w_h,r_h)\in V_h\times Q_h} \frac{A_h((v_h,q_h);(w_h,r_h))}{|||(v_h,q_h)|||\,|||(w_h,r_h)|||} \ge \beta.$$

M., Tobiska, 2015

Consistency

weak consisteny

$$\left|A_h((u-u_h,p-p_h);(w_h,r_h)\right| = |S_h(u,w_h)|$$

$$|S_h^a(u,w_h)| \le C \left(\sum_{M\in\mathcal{M}_h} \tau_M \, \|b\|^2_{0,\infty,M} \, h_M^{2s_1} \, \|u\|^2_{s_1+1,M} \right)^{1/2} \quad |||(w_h,r_h)|||$$

$$|S_h^b(u,w_h)| \le C \left(\sum_{M\in\mathcal{M}_h} \gamma_M \, h_M^{2s_3} \, \|u\|^2_{s_3+1,M} \right)^{1/2} \quad |||(w_h,r_h)|||$$

optimal order $\mathcal{O}(h^r)$ for $\tau_M \lesssim h_M^{2(r-s_1)}$ and $\gamma_M \lesssim h_M^{2(r-s_3)}$

M., Tobiska, 2015

Key in analysis for S_h^a

orthogonality $(q - i_h q, \varphi_h)_M = 0$ for all $\varphi_h \in D_2(M)$

usage: estimate of velocity-pressure coupling for stabilising term S_h^a

$$|(p - i_h p, \operatorname{div} w_h)_M| = |(p - i_h p, \operatorname{div} w_h - \pi_M^2 \operatorname{div} w_h)_M| = |(p - i_h p, \kappa_M^2 \operatorname{div} w_h)_M|$$

$$\Rightarrow \quad |(p - i_h p, \operatorname{div} w_h)| \le C \left(\sum_{M \in \mathcal{M}_h} \frac{h_M^{2r}}{\gamma_M} \|p\|_{r,M}^2 \right)^{1/2} |||(w_h, r_h)|||$$

orthogonality satisfied for

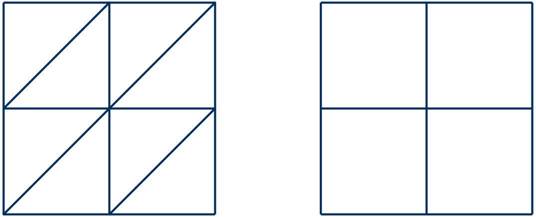
- $D_2(M) = \{0\}$
- $D_2(M) \subset (Q_h + \operatorname{span}(1))\big|_M$ (for discontinuous pressure)
- $D_2(M) \subset (Q_h + \operatorname{span}(1))\big|_M \cap H_0^1(M)$ (for continuous pressure)
  (bubble part of local pressure space)

Numerical results

prescribed solution of problem on $(0,1)^2$ with $\nu = 10^{-8}$, convection field $b = u$, $\sigma = 1$

$$u(x,y) = \begin{pmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{pmatrix}, \qquad p(x,y) = 2\cos(x)\sin(y) - p_0 \in L_0^2(\Omega)$$

coarsest meshes



one-level approach: $\mathcal{M}_h = \mathcal{T}_h$

Taylor–Hood family

simplices: $V_h = P_r, Q_h = P_{r-1}, r \geq 2$

$$D_1(K) = P_{s-1}(K),\ s \leq r, \qquad \tau_K \lesssim h_K^{2(r-s)},$$
$$D_2(K) = P_{t-1}(K),\ t \leq r - d - 1, \qquad \mu_K \sim 1$$

quadrilaterals/hexahedra: $V_h = Q_r, Q_h = Q_{r-1}, r \geq 2$

$$D_1(K) = Q_{s-1}(K),\ s \leq r, \qquad \tau_K \lesssim h_K^{2(r-s)},$$
$$D_2(K) = Q_{t-1}(K),\ t \leq r - 2, \qquad \mu_K \sim 1$$

convergence order: $\mathcal{O}(h^r)$

Taylor–Hood element P_3/P_2 on triangles

$$r = 3,\ d = 2 \quad \Longrightarrow \quad \begin{cases} s \le 3 & \Longrightarrow \quad D_1(K) \subset P_2(K), \qquad \tau_K \lesssim h_K^{6-2s} \\ t \le 0 & \Longrightarrow \quad D_2(K) \subset \{0\}, \qquad \mu_K \sim 1 \end{cases}$$

streamline			divergence			convergence	
s	$D_1(K)$	τ_K	t	$D_2(K)$	μ_K	error	order
3	$P_2(K)$	1	0	$\{0\}$	1	7.911-08	2.98
2	$P_1(K)$	h_K^2	0	$\{0\}$	1	7.694-08	2.99
1	$P_0(K)$	h_K^4	0	$\{0\}$	1	7.690-08	3.00
0	$\{0\}$	h_K^6	0	$\{0\}$	1	7.673-08	2.98
2	$P_2(K)$	1	1	$P_0(K)$	1	3.890-07	2.08

Discontinuous pressure

simplices: $V_h = P_r^+, Q_h = P_{r-1}^{\text{disc}}, r \geq 2$

$$D_1(K) = P_{s-1}(K),\ s \leq r, \qquad \tau_K \lesssim h_K^{2(r-s)},$$
$$D_2(K) = P_{t-1}(K),\ t \leq r, \qquad \mu_K \sim 1$$

quadrilaterals/hexahedra: $V_h = Q_r, Q_h = P_{r-1}^{\text{disc}}, r \geq 2$

$$D_1(K) = P_{s-1}(K),\ s \leq r, \qquad \tau_K \lesssim h_K^{2(r-s)},$$
$$D_2(K) = P_{t-1}(K),\ t \leq r, \qquad \mu_K \sim 1$$

convergence order: $\mathcal{O}(h^r)$

Element Q_3/P_2^{disc}

$$r = 3 \quad \Longrightarrow \quad \begin{cases} s \le 3 \quad \Longrightarrow \quad D_1(K) \subset P_2(K), & \tau_K \lesssim h_K^{6-2s} \\ t \le 3 \quad \Longrightarrow \quad D_2(K) \subset P_2(K), & \mu_K \sim 1 \end{cases}$$

streamline			divergence			convergence	
s	$D_1(K)$	τ_K	t	$D_2(K)$	μ_K	error	order
3	$P_2(K)$	1	3	$P_2(K)$	1	8.696-08	3.00
2	$P_1(K)$	h_K^2	2	$P_1(K)$	1	9.252-08	3.00
1	$P_0(K)$	h_K^4	1	$P_0(K)$	1	9.202-08	3.00
0	$\{0\}$	h_K^6	0	$\{0\}$	1	9.202-08	3.00

Summary

local projection stabilisation works for

- convection-diffusion equations
- Stokes, Oseen, Navier–Stokes equations
  equal-order case
  inf-sup stable discretisations
  LPS-norm as strong as SUPG/PSPG-like norms (Knobloch, Tobiska, 2015)

open questions

- precise choice of parameters for different problems
- two-level version of regularly refined tetrahedra