# A PDE approach to spectral fractional diffusion

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# Outline

# Outline

# Back to linear algebra I

- If $A$ is symmetric, that is
$$A^\intercal = A,$$
then it is diagonalizable.
- This means that there are $Q$ orthogonal, and $\Lambda$ diagonal, such that
$$A = Q^\intercal \Lambda Q, \quad Q^\intercal = Q^{-1}, \quad \Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}.$$
- In this case, the action $\mathbf{w} = A\mathbf{v}$ can be described as follows:
  - $\tilde{\mathbf{v}} = Q\mathbf{v}$ is a change of basis.
  - $\bar{\mathbf{v}} = \Lambda\tilde{\mathbf{v}}$ is a scaling in this new basis.
  - $\mathbf{w} = Q^\intercal \Lambda \bar{\mathbf{v}}$ is returning to the original basis.
- If, in addition, $A$ is positive, that is
$$\mathbf{v}^\intercal A\mathbf{v} > 0,$$
then *all its eigenvalues are positive* $\lambda_i > 0$.

# Back to linear algebra II

Why do we care about this? If $A \in \mathbb{R}^{n\times n}$ is symmetric:

- With this we can define almost any function of a matrix via

$$f(A) = Q^\intercal f(\Lambda) Q, \qquad f(\Lambda) = \mathrm{diag}\{f(\lambda_1), \dots, f(\lambda_n)\}.$$

- Solution of ODEs:

$$\dot{\mathbf{y}}(t) = A\mathbf{y},\ t > 0 \quad \mathbf{y}(0) = \mathbf{y}_0 \qquad \Longrightarrow \qquad \mathbf{y}(t) = \exp(tA)\mathbf{y}_0.$$

- Theory of iterative schemes: To solve $A\mathbf{x} = \mathbf{f}$ we can use a two-layer implicit scheme

$$B\frac{\mathbf{x}^{k+1} - \mathbf{x}^k}{\alpha} + A\mathbf{x}^k = \mathbf{f}$$

  with SPD preconditioner $B$. The analysis of such schemes can be reduced to that of the explicit one

$$\frac{\mathbf{v}^{k+1} - \mathbf{v}^k}{\alpha} + C\mathbf{v}^k = \mathbf{g}$$

  where

$$\mathbf{v}^k = B^{1/2}\mathbf{x}^k, \quad C = B^{-1/2}AB^{-1/2}, \quad \mathbf{g} = B^{-1/2}\mathbf{f}.$$

- ...

# Spectral theory 101

Question: What happens in infinite dimensions? In particular, for differential operators?

A (the?) basic partial differential operator that expresses diffusion is the *Laplacian*

$$-\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

- Integration by parts shows that $-\Delta$ is positive

$$\int_\Omega -\Delta v v \, \mathrm{d}x = \int_\Omega |\nabla v|^2 \, \mathrm{d}x > 0, \quad \forall v \in C_0^\infty(\Omega).$$

- One can show that $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact:
  - There exist $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \times L^2(\Omega)$ such that:

    $$-\Delta\varphi_k = \lambda_k\varphi_k, \qquad \varphi_{k|\partial\Omega} = 0$$

    and $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$.
  - This means that if $w \in L^2(\Omega)$, then it has the following representation

    $$w = \sum_{k=1}^{\infty} w_k\varphi_k \qquad w_k = \int_\Omega w\varphi_k \, \mathrm{d}x.$$

# The spectral fractional Laplacian I

- In addition, if $w$ is sufficiently nice, then we have that

$$-\Delta w = \sum_{k=1}^{\infty} w_k \lambda_k \varphi_k, \quad w_k = \int_\Omega w \varphi_k \, \mathrm{d}x$$

  which is an analogue of the matrix case:
  - The term $w_k$ is a change of basis.
  - Multiplication by the eigenvalue $\lambda_k$ is a diagonal scaling.
  - The outer sum is returning to the original basis.
- We can now define functions of $-\Delta$. For instance, if $s \in (0,1)$ and $w$ is sufficiently nice,

$$(-\Delta)^s w = \sum_{k=1}^{\infty} w_k \lambda_k^s \varphi_k,$$

Questions: Why do we care? What is the domain of this operator? What is its range?

# The spectral fractional Laplacian II

- The heat equation

$$\partial_t u - \Delta u = 0, \qquad u_{|t=0} = v$$

smoothens and smears the initial condition $v$. This could be used, for instance, in image denoising. However, the effect of $-\Delta$ is too strong. Thus, it can be weakened by

$$\partial_t u + (-\Delta)^s u = 0, \qquad u_{|t=0} = v.$$

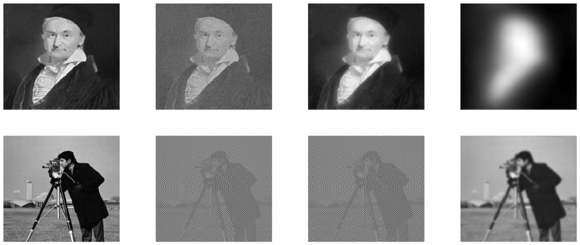
- Some special cases of random walks also lead to the fractional heat equation[1].
- Models in phase transition[2]: fractional Allen Cahn ($\alpha = 0$, $\beta \in (0,1)$) and Cahn Hilliard ($\alpha, \beta \in (0,1)$) equations

$$\partial_t u + (-\Delta)^\alpha \left(\varepsilon^2 (-\Delta)^\beta u + F'(u)\right) = 0,$$

[1] Valdinoci 2017

[2] Ainsworth and Mao 2017, Antil and Bartels 2018

# The spectral fractional Laplacian III



- Original, noisy, regularized images for $L^2$ and $H^{-1}$ fidelity terms.
- Top: $s = 0.42$
- Bottom: $s = 0.35$
- Stolen from 📘.

📘 Antil, Bartels 2017

# Spectral theory 102

- Let $\mathcal{L}$ be a symmetric second order elliptic operator, i.e.,

$$\mathcal{L}w = -\nabla\cdot(a\nabla w) + cw$$

  with $a \in L^\infty(\Omega, \mathbb{S}^d_+)$ uniformly positive definite and $0 \leq c \in L^\infty(\Omega)$.
- In a similar way we can define $\mathcal{L}^s_0$, the fractional powers of $\mathcal{L}$ supplemented with homogeneous Dirichlet (or Neumann) boundary conditions.
- From now on, and for simplicity only, we will only deal with the Laplacian. Everything that we will say applies to $\mathcal{L}^s_0$.

# Goal

- Given a suitable $f$ find $u$ such that

$$(-\Delta)^s u = f$$

  in the sense described above.
- Where's the catch? The domain $\Omega$ can be quite general, so the spectrum of $-\Delta$ is not readily available.

# Domain, range, and regularity I

- Because of the way that we defined the fractional Laplacian we have

$$(-\Delta)^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$$

  where

$$\mathbb{H}^s(\Omega) = \left\{ w = \sum_{k=1}^{\infty} w_k \varphi_k : \sum_{k=1}^{\infty} \lambda_k^s |w_k|^2 < \infty \right\}$$

- It turns out that

$$\mathbb{H}^s(\Omega) = \begin{cases} H^s(\Omega), & s \in \left(0, \frac{1}{2}\right), \\ H_{00}^{1/2}(\Omega), & s = \frac{1}{2}, \\ H_0^s(\Omega), & s \in \left(\frac{1}{2}, 1\right), \end{cases}$$

  where the zero subindices mean "zero boundary values".

- The fact that the domain has fractional Sobolev regularity reinforces the idea that we are taking fractional order derivatives.

# Domain, range, and regularity II

If we wish to develop a rigorous numerical approximation of $u$, then we must understand its regularity.

- From the definition it follows that, if $f \in \mathbb{H}^r(\Omega)$, then $u \in \mathbb{H}^{r+2s}(\Omega)$, for all $r \in \mathbb{R}$.
- If $r \geq -s$ this means that, at least for $\omega \Subset \Omega$,

$$u \in H^{r+2s}(\omega).$$

- What about near the boundary? For $x \in \bar{\Omega}$ let $\operatorname{dist}(x, \partial\Omega)$ be the distance of $x$ to $\partial\Omega$:
  - If $s \neq \frac{1}{2}$ then[1] there is a smooth function $v$ such that

$$u(x) \approx v(x) + \operatorname{dist}(x, \partial\Omega)^{\min\{1,2s\}}$$

  - If $s = \frac{1}{2}$ then we have the exceptional case[2]

$$u(x) \approx v(x) + \operatorname{dist}(x, \partial\Omega) \left|\log \operatorname{dist}(x, \partial\Omega)\right|.$$

[1] Caffarelli, Stinga 2016

[2] Costabel, Dauge 1993

# Outline

Motivation: Fractional powers of an operator

Direct discretization approach

Best uniform rational approximation

The Balakrishnan formula

The Caffarelli-Silvestre extension

# Direct discretization

Given $f \in \mathbb{H}^{-s}(\Omega)$,

$$f = \sum_{k=1}^{\infty} f_k \varphi_k : \quad (-\Delta)^s u = f \Longrightarrow \quad u_k = f_k \lambda_k^{-s}$$

Algorithm:

- Compute a "*sufficiently large*" number of eigenpairs $\{\lambda_k, \varphi_k\}_{k=1}^N$.
- Compute the Fourier coefficients $f_k$.
- Find the solution: $u_k = f_k \lambda_k^{-s}$.

But

- How to choose $N$?
- VERY time consuming!
- Error analysis?

# Error analysis I

The eigenpairs can only be computed approximately (read, via finite elements). The error analysis in this case is as follows[1]:

- Let $X$ be a Hilbert space and $A$ be a positive definite self-adjoint operator on $X$.
- Let $\{X_h\}_{h>0}$ be a family of closed subspaces of $X$ and $A_h$ is a positive definite bounded self-adjoint operator on $X_h$.
- Inverse estimate: There is $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{h \to 0} \varepsilon(0) = 0$ such that

$$\|A_h\| \lesssim \frac{1}{\varepsilon(h)}$$

- Approximability: If $P_h$ is the orthogonal projection onto $X_h$

$$\|(A_h^{-1} P_h - A^{-1}) f\|_X \lesssim \varepsilon(h) \|f\|_X$$

- In this case, for $s \in (0,1)$, we have

$$\|(A_h^{-s} P_h - A^{-s}) f\|_X \lesssim \varepsilon(h)^s \|f\|_X$$

[1] Matsuki, Ushijima 1993

# Error analysis II

In our case:

- $X = L^2(\Omega)$, $X_h$ is a (piecewise linear) finite element space, $A = -\Delta$, and $A_h = -\Delta_h$.
- Since $X_h$ consists of piecewise polynomials

$$\|A_h\| \lesssim \frac{1}{h^2}, \qquad \Longrightarrow \qquad \varepsilon(h) = h^2.$$

- For $f \in L^2(\Omega)$ we have

$$u = (-\Delta)^{-1} f \in H^2(\Omega) \cap H^1_0(\Omega)$$

  and, if $u_h \in X_h$ is its finite element approximation: $u_h = (-\Delta_h)^{-1} P_h f$, then Aubin–Nitsche duality yields

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^2 |u|_{H^2(\Omega)} \lesssim h^2 \|f\|_{L^2(\Omega)}$$

- The previous theory then gives

$$\|(-\Delta)^{-s} f - (-\Delta_h)^{-s} P_h f\|_{L^2(\Omega)} \lesssim h^{2s} \|f\|_{L^2(\Omega)}.$$

We still need to compute $(-\Delta_h)^{-s}$!

# Outline

# Computing the discrete spectrum

Evaluating the eigenvalues of $-\Delta_h$ is time consuming: MTT, Lanczos, ...
Best uniform rational approximation (BURA)[📕]: Assume we need to solve

$$\mathcal{A}^s \mathbf{u} = \mathbf{f}$$

where $\mathcal{A}$ is a rescaled version of $(-\Delta_h)^s$ so that its spectrum lies in $(0, 1]$.

- Let $r_s$ be analytic on $(0, 1]$ and, for some constant $\varepsilon > 0$ satisfies

$$\sup_{t \in (0,1]} |r_s(t) - t^{1-s}| \leq \varepsilon,$$

  then, for every $\gamma \in \mathbb{R}$ and $\mathbf{F} \in \mathbb{R}^N$ we have

$$\|(r_s(\mathcal{A}) - \mathcal{A}^{1-s})\mathbf{F}\|_{\mathcal{A}^\gamma} \leq \varepsilon \|\mathbf{F}\|_{\mathcal{A}^\gamma}$$

- The previous result implies that, if $\mathbf{u}_r = r_s(\mathcal{A})\mathcal{A}^{-1}\mathbf{f}$, then

$$\|\mathbf{u}_r - \mathbf{u}\|_{\mathcal{A}^\gamma} \leq \varepsilon \|\mathbf{f}\|_{\mathcal{A}^{-1}}$$

- Taking into account the discretization error, then $(\gamma = 0)$

$$\|u - u_{h,r}\|_{L^2(\Omega)} \lesssim h^{2s} + \varepsilon.$$

- Question: What is a suitable $r_s$?

📕 Harizanov, Lazarov, Margenov, Vutov 2016

# BURA

- We choose $r_s$ as the best uniform $(m,k)$-approximation to $t^{1-s}$.
- Apply a partial fraction decomposition to $t^{-1}r_s(t)$:

$$t^{-1}r_s(t) = \sum_{j=0}^{m-k-1} b_j t^j + \frac{c_0}{t} + \sum_{j=1}^{p_1} \frac{c_j}{t-d_j} + \sum_{j=1}^{p_2} \frac{B_j t + C_j}{(t-F_j)^2 + D_j^2}$$

where $k = p_1 + 2p_2$.
- To compute $\mathbf{u}_r = \mathcal{A}^{-1} r_s(\mathcal{A})\mathbf{f}$ we need to evaluate

$$\mathcal{A}^{-1} r_s(\mathcal{A})\mathbf{f} = \sum_{j=0}^{m-k-1} b_j \mathcal{A}^j \mathbf{f} + c_0 \mathcal{A}^{-1}\mathbf{f} + \sum_{j=1}^{p_1} c_j (\mathcal{A} - d_j \mathcal{I})^{-1}\mathbf{f}$$
$$+ \sum_{j=1}^{p_2} (B_j \mathcal{A} + C_j \mathcal{I})((\mathcal{A} - F_j \mathcal{I})^2 + D_j^2 \mathcal{I})^{-1}\mathbf{f}$$

- How do we choose $m$ and $k$? This is classical in rational approximation. For the optimal choice we have $m = k$ and

$$\varepsilon \lesssim 4^{2-s} |\sin \pi(1-s)| e^{-2\pi\sqrt{(1-s)k}}$$

so that, for this choice, the error decays exponentially in the polynomial degree.

# Outlook

To solve

$$(-\Delta)^s u = f$$

with BURA we must:

- Solve $\mathcal{O}(|\log h|)$ problems of the type $(-\Delta_h + c\mathcal{I})\mathbf{w} = \mathbf{g}$.
- Embarrassingly parallelizable.
- Error estimate
$$\|u - u_{h,r}\|_{L^2(\Omega)} \lesssim h^{2s}.$$

Questions:

- Other norms?
- Other types of problems? Time-dependent? Nonlinear?
- Stability? It is known that rational approximations are very sensitive to numerical rounding.

# Outline

# The Balakrishnan formula

- Notice that, for $\lambda > 0$ and $\theta \in (0,1)$

$$\frac{\sin \pi\theta}{\pi} \int_0^\infty t^{\theta-1}(\lambda + t)^{-1}\, \mathrm{d}t = \lambda^{\theta-1}.$$

- Functional calculus then says that, if $X$ is a Hilbert space and $A$ is a self-adjoint and positive operator on $X$:

$$A^\theta = AA^{\theta-1} = A\frac{\sin \pi\theta}{\pi} \int_0^\infty t^{\theta-1}(A + t\mathcal{I})^{-1}\, \mathrm{d}t.$$

- Let $X = L^2(\Omega)$ and $A = -\Delta$, then

$$\begin{aligned}
(-\Delta)^{-s} &= (-\Delta)^{-1}(-\Delta)^{1-s} \\
&= (-\Delta)^{-1}(-\Delta)\frac{\sin \pi(1-s)}{\pi} \int_0^\infty t^{1-s-1}(t\mathcal{I} - \Delta)^{-1}\, \mathrm{d}t \\
&= \frac{\sin \pi s}{\pi} \int_0^\infty t^{-s}(t\mathcal{I} - \Delta)^{-1}\, \mathrm{d}t
\end{aligned}$$

where we used the previous formula with $\theta = 1 - s$.

# Numerical scheme

Using

$$(-\Delta)^{-s} = \frac{\sin \pi\theta}{\pi} \int_0^\infty t^{-s}(t\mathcal{I} - \Delta)^{-1}\, \mathrm{d}t,$$

we can formulate the following game plan to devise a numerical scheme[📕]:

- Step 1: Use a quadrature for the $t$ variable:

$$(-\Delta)^{-s} f \approx \frac{\sin \pi s}{\pi} k \sum_{j=0}^{J} t_j^{-s}(t_j\mathcal{I} - \Delta)^{-1} f$$

- Step 2: Use standard finite element methods on the same mesh to approximate

$$w_j \in H_0^1(\Omega): \qquad t_j w_j - \Delta w_j = f \quad \text{in} \quad \Omega,$$

  i.e., $w_j = (t_j\mathcal{I} - \Delta)^{-1} f$.

- Step 3: Gather all contributions.

[📕] Bonito, Pasciak 2015

# Step 1: Sinc quadrature

- Change of variable: Let $t = e^y$ to get

$$u = (-\Delta)^{-s} f = \frac{\sin(\pi s)}{\pi} \int_{-\infty}^{\infty} e^{(1-s)y} (e^y I - \Delta)^{-1} f \, \mathrm{d}y.$$

- Quadrature: Given $N \in \mathbb{N}$, define $k = 1/\sqrt{N}$, $y_j = jk$ and the quadrature approximation

$$u^N = Q^N f = \frac{\sin(\pi s)}{\pi} k \sum_{j=-N}^{N} e^{(1-s)y_j} (e^{y_j} I - \Delta)^{-1} f.$$

- Exponential convergence: Let $s \in [0,1)$ and $r \in [0,1]$. If $f \in \mathbb{H}^r(\Omega)$, then

$$\|u - u^N\|_{\mathbb{H}^r(\Omega)} \lesssim e^{-c\sqrt{N}} \|f\|_{\mathbb{H}^r(\Omega)}.$$

# Steps 2 and 3: Finite element approximation and parallelization

- Let $X_h$ be a finite element space over $\Omega$, and assume that the mesh is quasiuniform.
- $w_h^j \in X_h$ are the finite element solutions of

$$(e^{y_j}\mathcal{I} - \Delta)w = f.$$

- These can be solved independently (embarrassingly parallelizable) and then gathered to obtain

$$u_h^N = \frac{\sin(\pi s)}{\pi} k \sum_{j=-N}^{N} e^{(1-s)y_j} w_h^j$$

# Error analysis

For simplicity, assume that $\Omega$ is convex.

- For $r \leq 2s$ define

$$\alpha_\star = \frac{1}{2}\left(\alpha + \min\{1-r,\alpha\}\right), \qquad \sigma = \max\{2\alpha_\star - 2s, 0\}.$$

  If $f \in \mathbb{H}^\sigma(\Omega)$ then

$$\|u - u_h^N\|_{\mathbb{H}^r(\Omega)} \lesssim h^{2\alpha_\star}|\log h|\|f\|_{\mathbb{H}^\sigma(\Omega)}.$$

- Setting $r = s$ we get

$$\|u - u_h^N\|_{\mathbb{H}^s(\Omega)} \lesssim h^{2-s}\|f\|_{\mathbb{H}^{2-2s}(\Omega)},$$

  which is "optimal" in order $2-s$ and regularity $f \in \mathbb{H}^{2-2s}(\Omega)$. However, this requires $u \in \mathbb{H}^2(\Omega)$, which is not generic!

# Outlook

To solve

$$(-\Delta)^s u = f$$

with the Balakrishnan formula we must:

- Solve $\mathcal{O}(|\log h|)$ problems of the type $(e^y \mathcal{I} - \Delta)w = f$.
- Embarrassingly parallelizable.
- Error estimate

$$\|u - u_h^N\|_{\mathbb{H}^s(\Omega)} \lesssim h^{2-s} \|f\|_{\mathbb{H}^{2-2s}(\Omega)},$$

Questions:

- Other types of problems? Time-dependent? Nonlinear?
- Lower regularity on $f$? How can we capture the boundary singularities of $u$?

# Outline

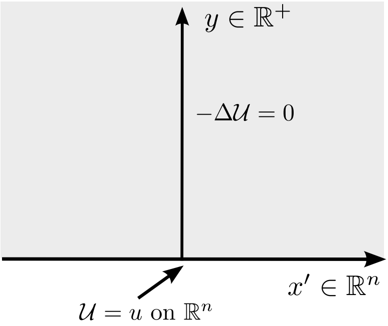
# $(-\Delta)^{1/2}$: The Dirichlet to Neumann operator I

- Let $u : \mathbb{R}^n \to \mathbb{R}$.
- Extend it harmonically to $\mathbb{R}^{n+1}_+$

$$-\Delta \mathcal{U} = 0, \text{ in } \mathbb{R}^{n_1}_+, \quad \mathcal{U}(\cdot, 0) = u$$

- The Dirichlet to Neumann map is

$$\mathrm{DtN} : u \mapsto -\partial_y \mathcal{U}(\cdot, 0).$$



# $(-\Delta)^{1/2}$: The Dirichlet to Neumann operator II

The Dirichlet to Neumann map

$$\mathrm{DtN} : u \mapsto -\partial_y \mathcal{U}(\cdot, 0).$$

has the following properties:

- $\mathrm{DtN}^2 = -\Delta$: Indeed, since $-\Delta_{x',y}\mathcal{U} = -\Delta_{x'}\mathcal{U} - \partial_y^2\mathcal{U} = 0$,

$$\mathrm{DtN}^2\, u = \partial_y \left(\partial_y \mathcal{U}(\cdot, 0)\right) = -\Delta_{x'}\mathcal{U}(\cdot, 0) = -\Delta_{x'} u.$$

- $\mathrm{DtN}$ is positive: Since $\mathcal{U}$ is harmonic

$$0 = -\int_{\mathbb{R}^{n+1}_+} \Delta\mathcal{U}\mathcal{U}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^{n+1}_+} |\nabla\mathcal{U}|^2\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^n} \partial_y\mathcal{U}\mathcal{U}\,\mathrm{d}x.$$

  On the other hand

$$\int_{\mathbb{R}^n} u\,\mathrm{DtN}\,u\,\mathrm{d}x = -\int_{\mathbb{R}^n} \partial_y\mathcal{U}\mathcal{U}\,\mathrm{d}x > 0.$$

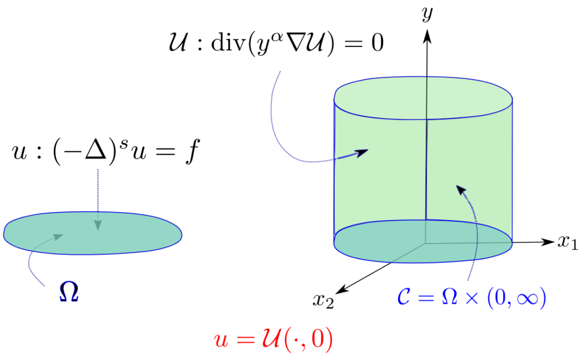
Thus, we define

$$\mathrm{DtN} = (-\Delta_x)^{\frac{1}{2}}, \quad (-\Delta_x)^{\frac{1}{2}} u = \partial_\nu \mathcal{U}.$$

# Outline

# The $\alpha$-harmonic extension I

The previous extension property can be generalized to any $s \in (0,1)$[📖]




- $s \in (0,1)$ and $\alpha = 1 - 2s \in (-1,1)$.
- $\partial_{\nu^\alpha} \mathcal{U} = -\lim_{y \downarrow 0} y^\alpha \partial_y \mathcal{U} = d_s f$ on $\Omega \times \{0\}$.
- $d_s = 2^\alpha \Gamma(1-s)/\Gamma(s)$.

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📖 Caffarelli, Silvestre 2007; Cabré, Tan 2010; Capella et al. 2011; Stinga, Torrea 2010–2012; Molčanov, Ostrovskiĭ 1969

# The $\alpha$-harmonic extension II

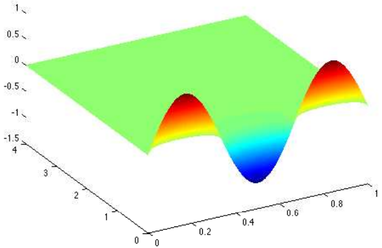
Fractional powers of $-\Delta$ can be realized as a generalization of the Dirichlet to Neumann operator:

$$\begin{cases} \partial_{yy}^2 \mathcal{U} + \frac{\alpha}{y}\partial_y \mathcal{U} + \Delta_x \mathcal{U} = 0 & \text{in } \mathcal{C} \\ \mathcal{U} = 0 & \text{on } \partial_L \mathcal{C} \\ \partial_{\nu^\alpha} \mathcal{U} = d_s f & \text{on } \Omega \times \{0\} \end{cases} \iff (-\Delta)^s u = f \text{ in } \Omega$$

$$u = \mathcal{U}(\cdot, 0).$$

Here:

- $\mathcal{C} = \Omega \times (0, \infty)$.
- $\alpha = 1 - 2s \in (-1, 1)$.
- $\partial_{\nu^\alpha} \mathcal{U} = -\lim_{y \downarrow 0} y^\alpha \partial_y \mathcal{U} = d_s f$.
- $d_s = 2^\alpha \Gamma(1-s)/\Gamma(s)$.



# The $\alpha$-harmonic extension III

Why does this make sense?

- For $\lambda > 0$ and $g \in \mathbb{R}$ consider the ODE:

$$\begin{cases} \psi'' + \dfrac{1-2s}{y}\psi' - \lambda\psi = 0, & \text{in } (0,\infty), \\ -\lim\limits_{y\downarrow 0} y^{1-2s}\psi' = d_s, & \lim\limits_{y\uparrow\infty}\psi(y) = 0. \end{cases}$$

- This is a Bessel equation with solution

$$\psi(y) = C_s\lambda^{-s}\left(\sqrt{\lambda}y\right)^s K_s(\sqrt{\lambda}y)$$

where $K_s$ is the modified Bessel function of the second kind.

- It is well known that $K_s(z) = az^{-s} + o(z^{-s})$, with $a > 0$ as $z \downarrow 0$. Thus

$$\psi(y) = c_s\lambda^{-s}\left(\sqrt{\lambda}y\right)^s\left(a(\sqrt{\lambda}y)^{-s}\right) \to ac_s\lambda^{-s}, \quad y \downarrow 0.$$

- Choosing $C_s$ appropriately we get $\psi(0) = \lambda^{-s}$.

# The $\alpha$-harmonic extension IV

- Recall that

$$f = \sum_{k=1}^{\infty} f_k \varphi_k \in \mathbb{H}^{-s}(\Omega), \quad (-\Delta)^s u = f, \quad \implies u = \sum_{k=1}^{\infty} \lambda_k^{-s} f_k \varphi_k$$

- Applying separation of variables to the extension problem[1]

$$u(x) = \sum_{k=1}^{\infty} u_k \varphi_k(x) \implies \mathcal{U}(x,y) = \sum_{k=1}^{\infty} u_k \varphi_k(x) \psi_k(y),$$

where the functions $\psi_k$ solve

$$\psi_k'' + \frac{\alpha}{y}\psi_k' = \lambda_k \psi_k, \text{ in } (0,\infty), \quad \psi_k(0) = 1, \quad \lim_{y\to\infty} \psi_k(y) = 0.$$

so that, as before,

$$\psi_k(y) = c_s \left(\sqrt{\lambda_k} y\right)^s K_s(\sqrt{\lambda_k} y),$$

[1] Capella et al. 2011

# Weak formulation

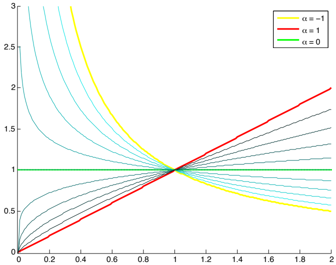
- Multiply $\nabla\cdot(y^\alpha \nabla \mathcal{U})$ by a test function $\phi$ and integrate over the cylinder $\mathcal{C}$ to obtain a possible weak formulation

$$\int_{\mathcal{C}} y^\alpha \nabla \mathcal{U} \cdot \nabla \phi \,\mathrm{d}x\,\mathrm{d}y = d_s \int_\Omega f \phi(x,0)\,\mathrm{d}x, \quad \forall \phi \in \mathring{H}^1_L(y^\alpha, \mathcal{C}),$$

- Where the energy space is

$$L^2(y^\alpha,\mathcal{C}) = \left\{ w : \int_{\mathcal{C}} |w|^2 y^\alpha \,\mathrm{d}x\,\mathrm{d}y < \infty \right\}$$

$$\mathring{H}^1_L(y^\alpha,\mathcal{C}) = \left\{ w \in L^2(y^\alpha,\mathcal{C}) : \ \nabla w \in L^2(y^\alpha,\mathcal{C}),\ w|_{\partial_L \mathcal{C}} = 0 \right\}.$$




The weight $y^\alpha$ is degenerate $(\alpha > 0)$ or singular$(\alpha < 0)$!

# Muckenhoupt weights

For every $a, b \in \mathbb{R}$, with $a < b$,

$$\frac{1}{b-a}\int_a^b |y|^\alpha \, \mathrm{d}y \cdot \frac{1}{b-a}\int_a^b |y|^{-\alpha} \, \mathrm{d}y \lesssim 1$$

which means $y^\alpha$ belongs to the Muckenhoupt class $A_2$.
This condition, essentially, means that $y^\alpha$ behaves like a constant at every scale!
Since $y^\alpha \in A_2$:

- The Hardy-Littlewood maximal operator is continuous on $L^2(y^\alpha, \mathcal{C})$.
- Singular integral operators are continuous on $L^2(y^\alpha, \mathcal{C})$.
- $L^2(y^\alpha, \mathcal{C}) \hookrightarrow L^1_{loc}(\mathcal{C})$.
- $H^1(y^\alpha, \mathcal{C})$ is Hilbert and $\mathcal{C}_b^\infty(\mathcal{C})$ is dense.
- Traces on $\partial_L \mathcal{C}$ are well defined.

# Weighted Sobolev spaces

- Weighted Poincaré inequality:

$$\int_{\mathcal{C}} y^{\alpha}|w|^2 \,\mathrm{d}x\,\mathrm{d}y \lesssim \int_{\mathcal{C}} y^{\alpha}|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}y \quad \forall w \in \mathring{H}^1_L(y^{\alpha}, \mathcal{C}).$$

- Surjective trace operator $\mathrm{tr}_{\Omega} : \mathring{H}^1_L(y^{\alpha}, \mathcal{C}) \to \mathbb{H}^s(\Omega)$.
- Lax-Milgram $\Rightarrow$ existence and uniqueness for every $f \in \mathbb{H}^{-s}(\Omega)$. Also

$$\|\mathcal{U}\|^2_{\mathring{H}^1_L(y^{\alpha},\mathcal{C})} = \|u\|^2_{\mathbb{H}^s(\Omega)} = d_s\|f\|^2_{\mathbb{H}^{-s}(\Omega)}.$$

We will discretize the $\alpha$-harmonic extension!

$$\mathcal{U} \in \mathring{H}^1_L(y^{\alpha}, \mathcal{C}) : \qquad \begin{cases} \nabla\cdot(y^{\alpha}\nabla\mathcal{U}) = 0 & \text{in } \mathcal{C} \\ \mathcal{U} = 0 & \text{on } \partial_L\mathcal{C} \\ \partial_{\nu^{\alpha}}\mathcal{U} = d_s f & \text{on } \Omega \times \{0\} \end{cases}$$

# Advantages and disadvantages

Advantages:

- Implementation requires standard numerical PDE components.
- It is very flexible as we will see later.

Disadvantages:

- One extra dimension! We have efficient solvers, and we will see later how to minimize the effect of $y$.
- Singular/degenerate weight $y^\alpha$? The weight $y^\alpha \in A_2$ for which there is a very well developed theory.

# Outline

# Solution representation

- Recall that we found, via separation of variables

$$u(x) = \sum_{k=1}^{\infty} \lambda_k^{-s} f_k \varphi_k(x) \implies \mathcal{U}(x,y) = \sum_{k=1}^{\infty} \lambda_k^{-s} f_k \varphi_k(x) \psi_k(y),$$

- The pairs $\{\lambda_k, \varphi_k\}_{k=1}^{\infty}$ are the eigenpairs of the Laplacian.
- The $\psi_k$ are

$$\psi_k(y) = c_s \left(\sqrt{\lambda_k} y\right)^s K_s(\sqrt{\lambda_k} y),$$

  where $K_s$ is the modified Bessel function of the second kind.
- The function $\psi_k$ satisfies, as $y \to \infty$,

$$\psi_k(y) \approx \left(\sqrt{\lambda_k} y\right)^{s-1/2} e^{-\sqrt{\lambda_k} y}.$$

- The function $\psi_k$ satisfies, as $y \to 0$,

$$\psi_k'(y) \approx y^{-\alpha}, \quad \psi_k''(y) \approx y^{-\alpha-1},$$

# Global Sobolev Regularity

- Compatible data: Let $f \in \mathbb{H}^{1-s}(\Omega)$, which means that $f$ has a vanishing trace for $s < \frac{1}{2}$.
- Space regularity:

$$\|\Delta_x \mathcal{U}\|^2_{L^2(y^\alpha,\mathcal{C})} + \|\partial_y \nabla_x \mathcal{U}\|^2_{L^2(y^\alpha,\mathcal{C})} = d_s \|f\|^2_{\mathbb{H}^{1-s}(\Omega)}$$

- Regularity in extended variable $y$: If $s \neq \frac{1}{2}$ and $\beta > 2\alpha + 1$ then

$$\|\partial_{yy}\mathcal{U}\|_{L^2(y^\beta,\mathcal{C})} \lesssim \|f\|_{L^2(\Omega)}.$$

  If $s = \frac{1}{2}$, then

$$\|\mathcal{U}\|_{H^2(\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1/2}(\Omega)}.$$

- Elliptic pick-up regularity: If $\Omega$ convex, then

$$\|w\|_{H^2(\Omega)} \lesssim \|\Delta_x w\|_{L^2(\Omega)} \quad \forall w \in H^2(\Omega) \cap H^1_0(\Omega).$$

  Under this assumption, we further have

$$\|D^2_x \mathcal{U}\|_{L^2(y^\alpha,\mathcal{C})} \lesssim \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

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Nochetto, Otárola, AJS 2015

# Analytic Regularity

- Behavior of $\psi(z) = c_s z^s K_s(z)$ near $z = 0$:

$$\left| \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} \psi(z) \right| \le C d_s \ell! z^{2s-\ell},$$

where $d_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$.

- Behavior of $\psi(z)$ for $z$ large:

$$\left| \frac{\mathrm{d}^\ell}{\mathrm{d}z^\ell} \psi(z) \right| \le C_{\epsilon,s} \ell! \epsilon^{-\ell} z^{s-\ell-\frac{1}{2}} e^{-(1-\epsilon)z}$$

- Global regularity of $\mathcal{U}$: If $0 \le \tilde{\nu} < s$ and $0 \le \nu < 1+s$, then there exists $\kappa > 1$ such that

$$\begin{aligned} \|\partial_y^{\ell+1}\mathcal{U}\|_{L^2(\omega_{\alpha+2\ell-2\tilde{\nu},\gamma},\mathcal{C})} &\lesssim \kappa^{\ell+1}(\ell+1)!\,\|f\|_{\mathbb{H}^{-s+\tilde{\nu}}(\Omega)}, \\ \|\nabla_x\partial_y^{\ell+1}\mathcal{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} &\lesssim \kappa^{\ell+1}(\ell+1)!\,\|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}, \\ \|\Delta_x\partial_y^{\ell+1}\mathcal{U}\|_{L^2(\omega_{\alpha+2(\ell+1)-2\nu,\gamma},\mathcal{C})} &\lesssim \kappa^{\ell+1}(\ell+1)!\,\|f\|_{\mathbb{H}^{1-s+\nu}(\Omega)}, \end{aligned}$$

with weight $\omega_{\beta,\gamma}(y) = y^\beta e^{\gamma y}, 0 \le \gamma < 2\sqrt{\lambda_1}$.

Banjai, Melenk, Nochetto, Otárola, AJS, Schwab 2018

# Outline

# Domain truncation

The domain $\mathcal{C}$ is infinite. We need to consider a truncated problem.

## Theorem (exponential decay)

*For every* $\mathcal{Y} > 0$

$$\|\mathcal{U}\|_{\mathring{H}_L^1(y^\alpha,\Omega\times(\mathcal{Y},\infty))} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2}\|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

Let $v$ solve

$$\begin{cases} \nabla\cdot(y^\alpha \nabla v) = 0 & \text{in } \mathcal{C}_{\mathcal{Y}} = \Omega \times (0,\mathcal{Y}), \\ v = 0 & \text{on } \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega \times \{\mathcal{Y}\}, \\ \partial_{\nu^\alpha} v = d_s f & \text{on } \Omega \times \{0\}. \end{cases}$$

## Theorem (exponential convergence)

*For all* $\mathcal{Y} > 0$*,*

$$\|\mathcal{U} - v\|_{\mathring{H}_L^1(y^\alpha,\mathcal{C}_{\mathcal{Y}})} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/4}\|f\|_{\mathbb{H}^{-s}(\Omega)}.$$

# Finite element method I: Mesh

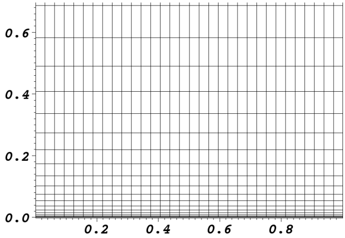
Let $\mathscr{T}_\Omega = \{K\}$ be triangulation of $\Omega$ (simplices or cubes)

- $\mathscr{T}_\Omega$ is conforming and shape regular.

Let $\mathscr{T}_\mathcal{Y} = \{T\}$ be a triangulation of $\mathcal{C}_\mathcal{Y}$ into cells of the form

$$T = K \times I, \quad K \in \mathscr{T}_\Omega, \quad I = (a, b).$$

$\mathcal{U}_{yy} \approx y^{-\alpha-1}$ as $y \approx 0+$ so we consider anisotropic elements




Shape regularity condition does NOT hold!

# Finite element method II: Spaces

We only require that if $T = K \times I$ and $T' = K' \times I'$ are neighbors

$$\frac{|I|}{|I'|} \approx 1,$$

This weak condition allows us to consider anisotropic meshes
Define

$$\mathbb{V}(\mathscr{T}_{\mathcal{Y}}) = \left\{ W \in \mathcal{C}^0(\bar{\mathcal{C}}_{\mathcal{Y}}) : \ W_{|T} \in \mathcal{P}_1(K) \otimes \mathbb{P}_1(I), \ W_{|\Gamma_D} = 0 \right\}$$

with $\Gamma_D = \partial_L \mathcal{C} \cup \Omega \times \{\mathcal{Y}\}$, and

$$\mathbb{U}(\mathscr{T}_{\Omega}) = \operatorname{tr}_{\Omega} \mathbb{V}(\mathscr{T}_{\mathcal{Y}}) = \left\{ W \in \mathcal{C}^0(\bar{\Omega}) : \ W_{|K} \in \mathcal{P}_1(K), \ W_{|\partial\Omega} = 0 \right\}.$$

Here $\mathcal{P}_1 = \mathbb{P}_1$ if $K$ is a simplex and $\mathcal{P}_1 = \mathbb{Q}_1$ if is a "brick".

# Finite element method III: Discrete problem

- Galerkin method for the extension: Find $V_{\mathscr{T}_\mathcal{Y}} \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ such that

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla V_{\mathscr{T}_\mathcal{Y}} \nabla W \, \mathrm{d}x \, \mathrm{d}y = d_s \int_\Omega f W(x,0) \, \mathrm{d}x, \quad \forall W \in \mathbb{V}(\mathscr{T}_\mathcal{Y}).$$

- Define

$$U_{\mathscr{T}_\Omega} = V_{\mathscr{T}_\mathcal{Y}}(\cdot, 0) \in \mathbb{U}(\mathscr{T}_\Omega).$$

- A trace estimate and Cèa's Lemma imply quasi-best approximation:

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim \|v - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})} = \inf_{W \in \mathbb{V}(\mathscr{T}_\mathcal{Y})} \|v - W\|_{\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})}$$

  We reduced the error analysis to a question of approximation theory in weighted spaces. Usually we set $W = \Pi v \in \mathbb{V}(\mathscr{T}_\mathcal{Y})$ where $\Pi$ is a suitable interpolation operator.

# The quasi-interpolation operator

We introduce an averaged interpolation operator $\Pi$[📕]

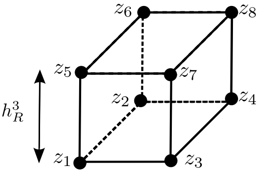
$$\Pi\phi(z) = Q_z^m\phi(z).$$

where $Q_z^m\phi$ is an averaged Taylor polynomial of $\phi$ of degree $m$.
Notice that:

- This is defined for all polynomial degree $m$ and any element shape (simplices or rectangles).
- We do not go back to the reference element — This is important for anisotropic estimates.

If the mesh is rectangular and Cartesian If $R$ and $S$ are neighbors

$$h_R^i/h_S^i \lesssim 1, \qquad i = \overline{1,N}.$$




[📕] Durán, Lombardi 2005; Dupont, Scott 1980; Sobolev 1950

# Error estimates on rectangles

### Theorem

*If* $\varpi \in A_p(\mathbb{R}^N)$, *and* $\phi \in W^1_p(\varpi, S_R)$

$$\|\phi - \Pi\phi\|_{L^p(\varpi,R)} \lesssim \sum_{i=1}^{N} h_R^i \|\partial_i \phi\|_{L^p(\varpi,S_R)}.$$

*If* $\phi \in W^2_p(\varpi, S_R)$

$$\|\partial_j(\phi - \Pi\phi)\|_{L^p(\varpi,R)} \lesssim \sum_{i=1}^{N} h_R^i \|\partial_i \partial_j \phi\|_{L^p(\varpi,S_R)},$$

$$\|\phi - \Pi\phi\|_{L^p(\varpi,R)} \lesssim \sum_{i,j=1}^{N} h_R^i h_R^j \|\partial_i \partial_j \phi\|_{L^p(\varpi,S_R)}.$$

- Directional estimates: note the products of the form $h_R^i h_R^j \|\partial_i \partial_j \phi\|_{L^p(\varpi,S_R)}$.
- Estimates on simplicial elements, different metrics and applications.

Nochetto, Otárola, AJS 2016

# Error estimates. Quasiuniform meshes

On quasiuniform meshes $h_T \approx h_K \approx h_I$ for all $T \in \mathscr{T}_\mathcal{Y}$, then

## Theorem (error estimates)

*The following estimate holds for all* $\epsilon > 0$

$$\begin{aligned} \|\nabla(v - V_{\mathscr{T}_\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})} &\lesssim h_K \|\partial_y \nabla_{x'} v\|_{L^2(y^\alpha, \mathcal{C})} + h_I^{s-\epsilon} \|\partial_{yy} v\|_{L^2(y^\beta, \mathcal{C})} \\ &\lesssim h^{s-\epsilon} \|f\|_{\mathbb{H}^{1-s}(\Omega)}. \end{aligned}$$

Consequently,

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim h^{s-\epsilon} \|f\|_{\mathbb{H}^{1-s}(\Omega)}.$$

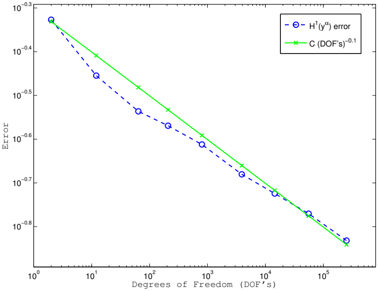
- This is suboptimal in terms of order (only order $s - \epsilon$)
- Is it sharp?

# Numerical experiment. Quasiuniform mesh

Let $\Omega = (0,1)$ and $f = \pi^{2s}\sin(\pi x)$, then

$$\mathcal{U} = \frac{2^{1-s}\pi^s}{\Gamma(s)}\sin(\pi x)y^s K_s(\pi y)$$

If $s = 0.2$, then




The energy error behaves like $DOFS^{-0.1} \approx h^{0.2}$, as predicted!

# Error estimates. Graded meshes

We use the principle of error equilibration. We use a graded mesh on $(0, \mathcal{Y})$

$$y_j = \mathcal{Y} \left( \frac{j}{M} \right)^{\gamma}, \quad j = \overline{0, M}, \quad \gamma > 1$$

$\mathcal{U}_{yy} \approx y^{-\alpha-1} \implies$ energy equidistribution for $\gamma > 3/(1-\alpha)$.

## Theorem (error estimates[1])

*If* $f \in \mathbb{H}^{1-s}(\Omega)$ *and* $\mathcal{Y} \approx |\log \# \mathscr{T}_{\mathcal{Y}}|$,

$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} = \|\nabla(\mathcal{U} - V_{\mathscr{T}_{\mathcal{Y}}})\|_{L^2(y^\alpha, \mathcal{C})} \lesssim |\log \# \mathscr{T}_{\mathcal{Y}}|^s \# \mathscr{T}_{\mathcal{Y}}^{-\frac{1}{n+1}} \|f\|_{\mathbb{H}^{1-s}(\Omega)},$$

*or equivalently*

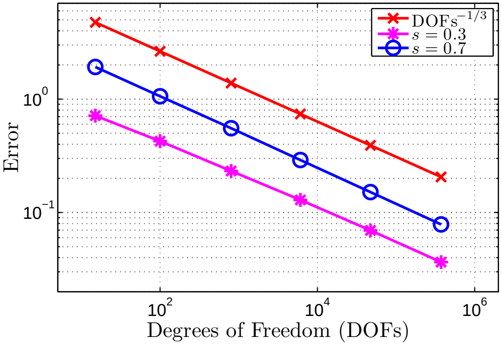
$$\|u - U_{\mathscr{T}_\Omega}\|_{\mathbb{H}^s(\Omega)} \lesssim |\log \mathscr{T}_\Omega|^s \mathscr{T}_\Omega^{-1/n} \|u\|_{\mathbb{H}^{1+s}(\Omega)}.$$

- This is near optimal in terms of regularity of $u \in \mathbb{H}^{1+s}(\Omega)$ and almost linear decay rate in $h$.
- This is suboptimal in terms of total number of degrees of freedom $\# \mathscr{T}_{\mathcal{Y}} \approx \# \mathscr{T}_\Omega^{1+\frac{1}{n}} \gg \# \mathscr{T}_\Omega$ with respect to the degrees of freedom in $\Omega$.

[1] Nochetto, Otárola, AJS 2015

# Numerical experiment

Experimental rates for circle and $s = 0.3$ and $s = 0.7$.
Set $\Omega = D(0,1) \subset \mathbb{R}^2$, $f = j_{1,1}^{2s} J_1(j_{1,1} r)(A_{1,1}\cos(\theta) + B_{1,1}\sin(\theta))$.
With graded meshes:




The experimental convergence rate $-1/3$ is optimal!

# Outline

# Diagonalization I

- Discretization in $y$: Let $\mathcal{G}^M$ be an arbitrary mesh in $(0,\mathcal{Y})$ with $M = \#\mathcal{G}^M$ and let $S^{\mathbf{r}}(0,\mathcal{Y};\mathcal{G}^M)$ be a FE space of polynomial degree $\mathbf{r}$ in $y$.
- Define
$$\mathbb{V}^{\mathbf{r}}_M(\mathcal{C}_{\mathcal{Y}}) = H^1_0(\Omega) \otimes S^{\mathbf{r}}(0,\mathcal{Y};\mathcal{G}^M).$$
  FE in $y$, continuous in $x$.
- Semidiscrete solution: $\mathcal{U}_M \in \mathbb{V}^{\mathbf{r}}_M(\mathcal{C}_{\mathcal{Y}})$ satisfies
$$\int_{\mathcal{C}_{\mathcal{Y}}} y^\alpha \nabla \mathcal{U}_M \nabla \phi \, \mathrm{d}x \, \mathrm{d}y = d_s \int_\Omega f \phi(x,0) \, \mathrm{d}x \quad \forall \phi \in \mathbb{V}^{\mathbf{r}}_M(\mathcal{C}_{\mathcal{Y}}).$$
- Exponential convergence: Let $f \in \mathbb{H}^{-s+\nu}(\Omega)$ for $0 < \nu < s$. If $\mathcal{Y} \approx M$, the mesh $\mathcal{G}^M$ is geometric towards $y = 0$, and the polynomial degree $\mathbf{r}$ grows linearly from $y = 0$, then there exists $b > 0$ such that
$$\|\nabla(\mathcal{U} - \mathcal{U}_M)\|_{L^2(y^\alpha,\mathcal{C})} \lesssim e^{-bM} \|f\|_{\mathbb{H}^{-s+\nu}(\Omega)}.$$

# Diagonalization II

- Eigenvalue problem: Let $\mathcal{M} = \dim S^{\mathbf{r}}(0,\mathcal{Y};\mathcal{G}^M)$ and $(\mu_i, v_i)_{i=1}^{\mathcal{M}}$ be the (normalized) eigenpairs of

$$\mu \int_0^{\mathcal{Y}} y^\alpha v'(y) w'(y)\,\mathrm{d}y = \int_0^{\mathcal{Y}} y^\alpha v(y) w(y)\,\mathrm{d}y \qquad \forall w \in S^{\mathbf{r}}(0,\mathcal{Y};\mathcal{G}^M).$$

- Representation: If $\mathcal{U}_M(x,y) = \sum_{j=1}^{\mathcal{M}} U_j(x) v_j(y)$ with $U_j \in H_0^1(\Omega)$, then

$$a_{\mu_i,\Omega}(U_i, V) = d_s v_i(0) \int_\Omega f V\,\mathrm{d}x \qquad \forall V \in H_0^1(\Omega),$$

where $a_{\mu_i,\Omega}$ are the singularly perturbed bilinear forms

$$a_{\mu_i,\Omega}(U,V) := \int_\Omega (\mu_i \nabla_x U \nabla_x V\,\mathrm{d}x + UV)\,\mathrm{d}x$$

# Tensor product discretization

- Ritz projections: $\Pi_i u \in S_0^q(\mathscr{T}_\Omega)$ satisfies

$$a_{\mu_i,\Omega}(u - \Pi_i u, v) = 0 \quad \forall\, v \in S_0^q(\mathscr{T}_\Omega),$$

  where $S_0^q(\mathscr{T}_\Omega) \subset H_0^1(\Omega)$ is the FE space of piecewise polynomials of degree $\leq q$ over $\mathscr{T}_\Omega$.

- Discrete solution: Let $U_{h,M} \in S_0^q(\mathscr{T}_\Omega) \otimes S^{\mathbf{r}}(0, \mathcal{Y}; \mathcal{G}^M)$ satisfy

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla U_{h,M} \nabla V \,\mathrm{d}x \,\mathrm{d}y = d_s \int_\Omega f V(x,0) \,\mathrm{d}x, \; \forall V \in S_0^q(\mathscr{T}_\Omega) \otimes S^{\mathbf{r}}(0, \mathcal{Y}; \mathcal{G}^M)$$

  and note that it can be represented as follows

$$U_{h,M}(x,y) = \sum_{i=1}^{\mathcal{M}} \Pi_i U_i(x) v_i(y).$$

- Parallelization: This corresponds to solving $\mathcal{M}$ decoupled elliptic problems with the singularly perturbed bilinear form $a_{\mu_i,\Omega}$ for $i = 1, \ldots, \mathcal{M}$.

# Tensor $\mathbb{P}_1$-FEM

- Assume that $f \in L^2(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is a polygon with corners $\mathbf{c}$.
- The solution to

$$-\Delta_x w = f, \text{ in } \Omega \quad w = 0, \text{ on } \partial\Omega \implies$$

$$\|w\|_{H^2_\beta(\Omega)} \lesssim \|f\|_{L^2(\Omega)}, \qquad |w|^2_{H^2_\beta(\Omega)} = \int_\Omega \prod_{\mathbf{c}} |x' - \mathbf{c}|^{2\beta} |D^2 w|^2 \, \mathrm{d}x.$$

- This type of singularity can be captured by using a graded mesh in $\Omega$: Let $\mathscr{T}_\Omega$ be graded towards the re-entrant corners so that, if $N = \#\mathscr{T}_\Omega$ and $h = N^{-1/2}$, for any $w \in S^1_0(\mathscr{T}_\Omega)$

$$N\|w - \Pi w\|^2_{L^2(\Omega)} \lesssim \|w\|^2_{H^1(\Omega)}, \quad N^2\|w - \Pi w\|^2_{L^2(\Omega)} \lesssim \|w\|^2_{H^2_\beta(\Omega)}.$$

- With this construction we obtain that, if $\mathcal{G}^M_\eta$ is a suitably graded radical mesh $\left\{y_i = \left(\frac{i}{M}\right)^\eta \mathcal{Y}\right\}_{i=0}^M$, with $\eta s > 1$ and $M \approx N^{\frac{1}{2}} = (\#\mathscr{T}_\Omega)^{\frac{1}{2}}$, the discrete solution $U_{h,M}$ satisfies

$$\|u - \operatorname{tr}_\Omega U_{h,M}\|_{\mathbb{H}^s(\Omega)} \leq h\|f\|_{\mathbb{H}^{1-s}(\Omega)}$$

and

$$\dim \mathbb{V}^{1,1}_{h,M}(\mathscr{T}_\Omega, \mathcal{G}^M) \approx h^{-3} \log|\log h| \approx N_\Omega^{1+\frac{1}{2}} \log\log N_\Omega.$$

# Sparse grid FEM

- Complexity of tensor product: $N_\Omega^{1+\frac{1}{2}}$ is suboptimal.
- To overcome this we use a sparse grid space. Let

$$\mathbb{V}_L^{1,1}(\mathcal{C}_\mathcal{Y}) = \sum_{\ell,\ell' \geq 0,\, \ell+\ell' \leq L} S_0^1(\mathscr{T}_\Omega^\ell) \otimes S^1(0,\mathcal{Y};\mathcal{G}_\eta^{2^{\ell'}}),$$

  where $\mathscr{T}_\Omega^\ell$ and $\mathcal{G}_\eta^{2^{\ell'}}$ are nested meshes of levels $\ell$ and $\ell'$ graded towards corners $\mathbf{c}$ of $\Omega$ and $y = 0$, respectively.

- We have the error estimate: Let $1 < \nu < 1 + s$, $\eta(\nu - 1) \geq 1$, and $\mathcal{Y} \approx |\log h_L|$. If $f \in \mathbb{H}^{-s+\nu}(\Omega)$, then $\mathcal{U}_L \in \mathbb{V}_L^{1,1}(\mathcal{C}_\mathcal{Y})$ satisfies

$$\|\mathcal{U} - \mathcal{U}_L\|_{L^2(y^\alpha,\mathcal{C})} \lesssim h_L |\log h_L| \, \|f\|_{\mathbb{H}^{-s+\nu}(\Omega)},$$
$$\dim \mathbb{V}_L^{1,1}(\mathcal{C}_\mathcal{Y}) \lesssim N_\Omega \log\log N_\Omega.$$

- The complexity of sparse grids is quasi-optimal in terms of $N_\Omega$.

# $hp$-FEM in $y$ and $\mathbb{P}_1$-FEM in $\Omega$

- Graded geometric mesh: Let $\mathcal{G}_\sigma^M = \{\mathcal{Y}\sigma^{M-i}\}_{i=1}^M$ with $\sigma < 1$.
- Data regularity: $f \in \mathbb{H}^{1-s}(\Omega)$ and $\Omega \subset \mathbb{R}^2$ is a polygon with corners $\mathbf{c}$.
- FE space: $\mathbb{V}_{h,M}^{1,\mathbf{r}}(\mathscr{T}_\Omega, \mathcal{G}_\sigma^M)$ is the space of piecewise polynomials of degree one over $\mathscr{T}_\Omega$ and piecewise polynomials of degree $\mathbf{r}$ growing linearly from $1$ over $\mathcal{G}_\sigma^M$.
- Error estimates: Let $\mathscr{T}_\Omega$ be a suitably graded mesh towards the re-entrant corners $\mathbf{c}$. If $\mathcal{Y} \approx |\log h|$ and $U_{h,M} \in \mathbb{V}_{h,M}^{1,\mathbf{r}}(\mathscr{T}_\Omega, \mathcal{G}_\sigma^M)$ is the Galerkin solution, then

$$\|\nabla(\mathcal{U} - U_{h,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim h\|f\|_{\mathbb{H}^{1-s}(\Omega)}$$

$$\dim \mathbb{V}_{h,M}^{1,\mathbf{r}}(\mathscr{T}_\Omega, \mathcal{G}_\sigma^M) \approx h^{-2}|\log h|^2 \approx N_\Omega |\log N_\Omega|$$

- Complexity: This is quasi-optimal in terms of $N_\Omega$.

# $hp$-FEM in $y$ and $\Omega$

- Data regularity: The domain $\Omega \subset \mathbb{R}^2$ and $f$ are analytic.
- Graded mesh in $\Omega$: The mesh $\mathscr{T}_\Omega$ is anisotropic and graded towards $\partial\Omega$ so that it resolves the smallest scale $\mu_{\mathcal{M}}$ of the singularly perturbed problems originating from the diagonalization.
- Graded mesh in $y$: Let $\mathcal{G}_\sigma^M = \{\mathscr{Y}\sigma^{M-i}\}_{i=1}^M$ with $\sigma < 1$.
- Error estimate: If $\mathscr{Y} \approx M$, $\mathbf{r}$ grows linearly from $y = 0$, then the Galerkin solution $U_{h,M} \in S_0^q(\mathscr{T}_\Omega) \otimes S^{\mathbf{r}}(\mathcal{G}_\sigma^M)$ and the total number $N_{\Omega,\mathscr{Y}}$ of degrees of freedom satisfy

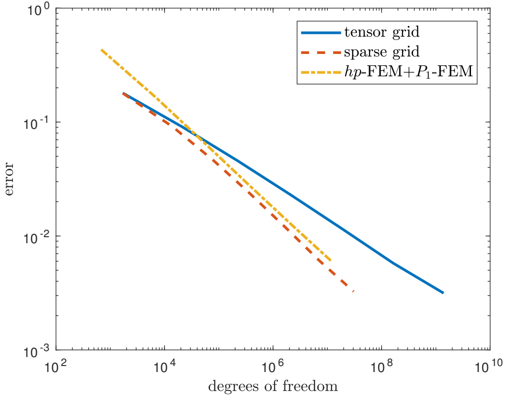
$$\|\nabla(\mathcal{U} - U_{h,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim M^2 e^{-bq} + e^{-bM}$$
$$N_{\Omega,\mathscr{Y}} \approx q^2 M^3.$$

- Exponential rate of convergence: If $q \approx M$, then

$$\|\nabla(\mathcal{U} - U_{h,M})\|_{L^2(y^\alpha,\mathcal{C})} \lesssim e^{-b' N_{\Omega,\mathscr{Y}}^{1/5}}.$$

# Numerical experiment. Performance of tensor FEMs

- Data: $\Omega$ L-shaped domain in $\mathbb{R}^2$; $f = 1$; $s = 3/4$.
- Error: It is always measured in the energy space $\mathbb{H}^s(\Omega)$.




- Conclusions: Both sparse grid FEM and $hp$-FEM reduced substantially the DOFs relative to tensor FEM and deliver quasi-optimal complexity.

# Outline

# Outlook I

- PDE approach. The extension converts the nonlocal problem into a local PDE problem in one higher dimension. This is very flexible:
  - Parabolic problems[1] Details.
  - Stationary[2] Details and time dependent[3] Details obstacle problems.
- We have a complete and quasi-optimal a priori error analysis over anisotropic meshes. The complexity, in terms of total degrees of freedom, is:
  - $\mathbb{P}_1 - \mathbb{P}_1$*-elements:* suboptimal complexity and linear rate for $\Omega$ convex and compatible data. Extension to non-convex domains.
  - *Sparse tensor* $\mathbb{P}_1 - \mathbb{P}_1$*-elements:* quasi-optimal complexity and linear rate for $\Omega$ polygonal with compatible data.
  - $hp$*-elements:* quasi-optimal complexity and exponential rate for analytic but incompatible data.
  - We also have multigrid methods[4] Details, a posteriori error estimators[5] Details.

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[1] Nochetto, Otárola, AJS 2016

[2] Nochetto, Otárola, AJS 2015

[3] Otárola, AJS 2016

[4] Chen, Nochetto, Otárola, AJS 2016

[5] Chen, Nochetto, Otárola, AJS 2015

# Outlook II

Questions:

- Adaptivity: Convergence and optimality is still open (issue is anisotropic meshes and lack of shape regularity).
- $3d$-computations: A virtual implementation of extended variable is open.
- Theory and implementation of $3d$ $hp$-FEM are open.

# Outline

# Outline

A posteriori error analysis
- Motivation
- A fundamental difficulty
- Anisotropic error estimation

Multilevel methods

Time dependent problems

Elliptic obstacle problem

Parabolic obstacle problem

# Adaptivity

Adaptivity is motivated by:

- Computational efficiency: extra $n+1$-dimension.
- The a priori theory requires:
  - Regularity of the datum: $f \in \mathbb{H}^{1-s}(\Omega)$.
  - Regularity of the domain: $\Omega$ is $C^{1,1}$ or a convex polygon.
- If one of these conditions is violated, the solution $\mathcal{U}$ may have singularities in $\Omega$ which lead to fractional regularity.
- Quasiuniform refinement of $\Omega$ would not result in an efficient solution technique.
- We need anisotropic a posteriori error estimators.

# Adaptive Loop

We consider an *almost* standard adaptive loop:

$$\text{SOLVE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE}$$

except for the statements in red below:

- **SOLVE**: Finds the Galerkin solution $V_{\mathscr{T}_\mathcal{Y}}$.
- **ESTIMATE**: Computes a star-indicator $\mathcal{E}_{z'}$ for every node $z' \in \Omega$.
- **MARK**: For $\theta \in (0,1)$ choose a minimal subset of nodes $\mathcal{M}$:

$$\mathcal{E}_{\mathcal{M}}^2 = \sum_{z' \in \mathcal{M}} \mathcal{E}_{z'}^2 \geq \theta^2 \mathcal{E}_{\mathscr{T}}^2.$$

- **REFINE**: Given a set of marked nodes $\mathcal{M}$
  - Refine the cells $K \ni z'$ for all $z' \in \mathcal{M}$ to get $\widetilde{\mathscr{T}_\Omega}$.
  - Create an anisotropic mesh $\{y_j\}_{j=1}^M$ so that grading $y_j = \mathcal{Y}\left(\frac{j}{M}\right)^\gamma$ holds.
  - The refined mesh is $\widetilde{\mathscr{T}_\mathcal{Y}} = \widetilde{\mathscr{T}_\Omega} \times \{\widetilde{I}\}$ with $\widetilde{I} = [y_{j-1}, y_j]$.

# Outline

# Isotropic a posteriori error indicators

- Residual error indicator: If we were to integrate by parts the discrete problem over an element $T \in \mathscr{T}_{\mathcal{Y}}$, we would get

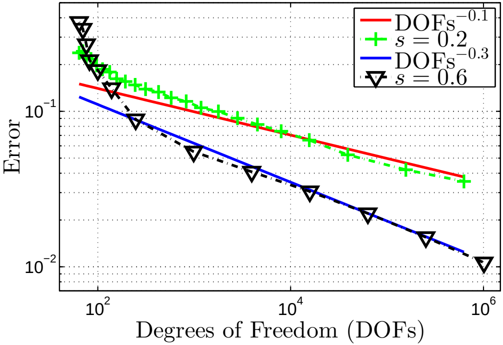
$$\int_T y^\alpha \nabla V \nabla W = \int_{\partial T} y^\alpha W \nabla V \cdot \boldsymbol{\nu} - \int_T \nabla \cdot (y^\alpha \nabla V) W$$

  Since $\alpha \in (-1,1)$, the boundary integral is meaningless for $y = 0$.
- Alternative error indicators: Residual indicators are not the only possibility:
  - Local problems on stars: $\mathcal{E}_z^2 = \int_{S_z} y^\alpha |\nabla Z|^2$ ($Z$ solution of a BVP in $S_z$).
  - Zienkiewicz-Zhu estimators.
  - Hypercircle estimators.
- Local problems on stars: We prove for all nodes $z \in \mathcal{N}$

$$\mathcal{E}_z^2 \lesssim \|\nabla(v - V)\|^2_{L^2(y^\alpha, S_z)} \lesssim \mathcal{E}_z^2 + \operatorname{osc}(y^\alpha, V, f, S_z)^2$$

# Numerical Experiment with Isotropic Refinement

- Set $\mathcal{C}_\mathcal{Y} = (0,1) \times (0,4)$ and $u = \sin(\pi x)$
- Experimental convergence rates:

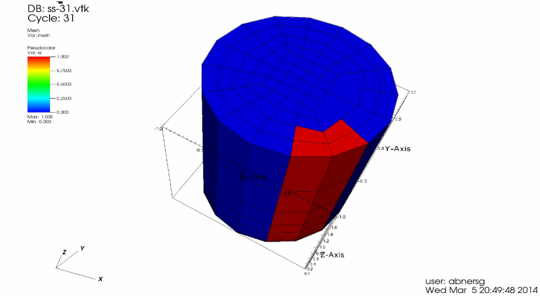



- The error decays like $(\#\mathscr{T}_\mathcal{Y})^{-(1-|\alpha|)/4}$ as in uniform/isotropic refinement!
- Does adaptivity help?

# Outline

# Anisotropic Error Estimation

- Anisotropic a posteriori error estimator: we need to distinguish the behavior on the extended variable $y$ from the rest.
- The theory of a posteriori error estimation (and adaptivity) on anisotropic discretizations is still in its infancy.
- Cylindrical stars: We propose an error estimator based on solving local problems on sets $\mathcal{C}_{z'} = S_{z'} \times (0, \mathcal{Y})$ as depicted in red in the figure:



# An Ideal A Posteriori Error Estimator

- Local space: For $z' \in \Omega$ a node, let $\mathcal{C}_{z'} = S_{z'} \times (0, \mathscr{Y})$ and define

$$\mathcal{W}(\mathcal{C}_{z'}) = \left\{ w \in H^1(y^\alpha, \mathcal{C}_{z'}) : \; w = 0 \text{ on } \partial\mathcal{C}_{z'} \setminus \Omega \times \{0\} \right\}.$$

- Local star indicator: The error indicator $\eta_{z'} \in \mathcal{W}(\mathcal{C}_{z'})$ is given by

$$\int_{\mathcal{C}_{z'}} y^\alpha \nabla \eta_{z'} \nabla w \, \mathrm{d}x \, \mathrm{d}y = d_s \int_\Omega f w(x,0) \, \mathrm{d}x - \int_{\mathcal{C}_{z'}} y^\alpha \nabla V \nabla w \, \mathrm{d}x \, \mathrm{d}y,$$

  for every $w \in \mathcal{W}(\mathcal{C}_{z'})$.

- Global error estimator:

$$\mathcal{E}_{\mathscr{T}_\Omega} = \left( \sum_{z'} \mathcal{E}_{z'}^2 \right)^{1/2}, \quad \mathcal{E}_{z'} = \|\nabla \eta_{z'}\|_{L^2(y^\alpha, \mathcal{C}_{z'})}.$$

# Anisotropic a posteriori error analysis

- Efficiency: For every node $z' \in \Omega$ we have

$$\mathcal{E}_{z'} \leq \|\nabla e\|_{L^2(y^\alpha, \mathcal{C}_{z'})}.$$

- Data oscillation: If $f_{z'|K} = \frac{1}{|K|}\int_K f \, \mathrm{d}x$ for every element $K \subset S_{z'}$, then

$$\operatorname{osc}_{\mathscr{T}_\Omega}(f)^2 = \sum_{z'} \operatorname{osc}_{z'}(f)^2, \quad \operatorname{osc}_{z'}(f)^2 = d_s h_{z'}^{2s} \|f - f_{z'}\|^2_{L^2(S_{z'})}$$

- Reliability:

$$\|\nabla e\|^2_{L^2(y^\alpha, \mathcal{C}_\mathscr{Y})} \lesssim \mathcal{E}^2_{\mathscr{T}_\Omega} + \operatorname{osc}_{\mathscr{T}_\Omega}(f)^2.$$

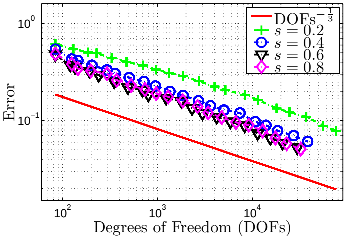
- Computable estimator: Restrict $\mathcal{W}(\mathcal{C}_{z'})$ to a discrete subspace

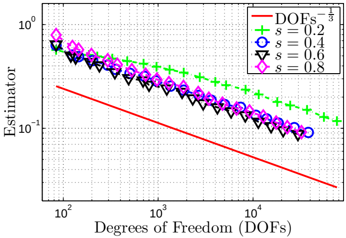
$$\{W \in \mathcal{W}(\mathcal{C}_{z'}) : W|_T \in \mathcal{P}_2(K) \otimes \mathbb{P}_2(I), \forall T = K \times I\}$$

$\mathcal{P}_2(K) = \mathbb{Q}_2(K)$ for rectangles, $\mathcal{P}_2(K) = \mathbb{P}_2(K) \oplus \mathbb{B}_3(K)$ for simplices.

# Numerical experiment I

- $\Omega$ is the standard L-shaped domain in 2d.
- $f = 1$ which, for $s < \frac{1}{2}$, is incompatible with the problem and creates a boundary layer.
- Experimental error and estimator: error computed against a very fine discrete solution.

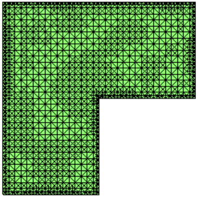


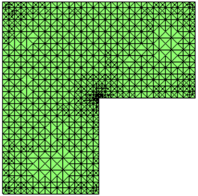

- Optimal decay rate: We get $DOF^{-1/3}$ for all $s$.

# Numerical experiment II: Meshes

- Meshes: For $s < 1/2$ the solution exhibits a boundary layer.



$s = 0.2$



$s = 0.8$

- Question: Is there any theory on anisotropic adaptive approximation[📕]?

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[📕] Cohen Mirebeau 2010–2012, Petrushev 2007–2009

# Outline

A posteriori error analysis

Multilevel methods

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# Multilevel methods

If you do not diagonalize, How do you solve the equations?
We use multilevel methods.

- We have a sequence of nested meshes $\mathscr{T}_0 \preceq \mathscr{T}_1 \preceq \cdots \preceq \mathscr{T}_J$ which induces a sequence of nested FE spaces

$$\mathbb{V}_0 \subset \mathbb{V}_1 \subset \cdots \subset \mathbb{V}_J = \mathbb{V}.$$

- Introduce the space macro and micro decomposition

$$\mathbb{V} = \sum_{k=0}^{J} \mathbb{V}_k = \sum_{k=0}^{J} \sum_{j=1}^{\mathcal{M}_k} \mathbb{V}_{k,j}.$$

- Define a multigrid algorithm as a standard SSC[📘] over this decomposition.
- This setting allows for point and line smoothers.

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📘 Xu 1992

# Properties of the decomposition

## Lemma (stability and inverse inequality)

*Let $v \in \mathbb{V}$ and $v = \sum_{i=1}^{\mathcal{N}} v_i$ be the line decomposition of $v$. Then we have the norm equivalence*

$$\sum_{i=1}^{\mathcal{N}} \|v_i\|^2_{L^2(y^\alpha,\mathcal{C})} \lesssim \|v\|^2_{L^2(y^\alpha,\mathcal{C})} \lesssim \sum_{i=1}^{\mathcal{N}} \|v_i\|^2_{L^2(y^\alpha,\mathcal{C})}.$$

*Moreover, for every $K \in \mathscr{T}_\Omega$ we have*

$$\|\nabla v\|_{L^2(y^\alpha,K\times(0,\mathcal{Y}))} \lesssim h_K^{-1}\|v\|_{L^2(y^\alpha,K\times(0,\mathcal{Y}))}.$$

*In both inequalities the hidden constant is independent of $J$ and depends on $y^\alpha$ only through $C_{2,y^\alpha}$.*

- The proof relies fundamentally on the fact that $y^\alpha \in A_2$.

# Convergence rate

## Theorem (convergence of multigrid)

*The contraction rate of the multigrid algorithm is*

$$\delta \leq 1 - \frac{1}{1 + CJ}$$

*where the constant $C$ is independent of the mesh size, and it depends on $y^\alpha$ only through $C_{2,y^\alpha}$.*

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Chen, Nochetto, Otárola, AJS 2015

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# Space-time fractional parabolic problem

Let $T > 0$ be some positive time. Given $f : \Omega \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$ find $u$ such that

$$\partial_t^\gamma u + (-\Delta)^s u = f \text{ in } \Omega \times (0,T] \quad u|_{t=0} = u_0 \text{ in } \Omega.$$

Here $\gamma \in (0,1]$.
For $\gamma = 1$ this is the usual time derivative, if $\gamma < 1$ we consider the Caputo derivative

$$\partial_t^\gamma u(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial_r u(x,r)}{(t-r)^\gamma} \, \mathrm{d}r = [I^{1-\gamma} \partial_r u(x,\cdot)](t),$$

where $I^\sigma$ is the *Riemann-Liouville* fractional integral of order $\sigma$.

Nonlocality in space and time!
We will overcome the nonlocality in space using the Caffarelli-Silvestre extension.

# Extended evolution problem

The Caffarelli-Silvestre extension turns our problem into a quasistationary elliptic problem with dynamic boundary condition

$$\begin{cases} -\nabla\cdot\left(y^{\alpha}\nabla\mathcal{U}\right)=0, & \text{in } \mathcal{C},\ t\in(0,T), \\ \mathcal{U}=0, & \text{on } \partial_L\mathcal{C},\ t\in(0,T), \\ d_s\partial_t^{\gamma}\mathcal{U}+\dfrac{\partial\mathcal{U}}{\partial\nu^{\alpha}}=d_sf, & \text{on } \Omega\times\{0\},\ t\in(0,T), \\ \mathcal{U}=\mathsf{u}_0, & \text{on } \Omega\times\{0\},\ t=0. \end{cases}$$

Connection: $\mathsf{u}=\mathcal{U}(x,0)$, $\alpha=1-2s$.

Nonlocality just in time!

Weak formulation: seek $\mathcal{U}\in\mathbb{V}$ such that for a.e. $t\in(0,T)$,

$$\begin{cases} \displaystyle\int_{\Omega}\partial_t^{\gamma}\mathcal{U}(x,0)\phi(x,0)\,\mathrm{d}x+a(w,\phi)=\int_{\Omega}f\phi(x,0)\,\mathrm{d}x, \\ \mathcal{U}_{|t=0}=\mathsf{u}_0 \end{cases}$$

for all $\phi\in\mathring{H}_L^1(y^{\alpha},\mathcal{C})$, where

$$a(w,\phi)=\frac{1}{d_s}\int_{\mathcal{C}}y^{\alpha}\nabla w\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}y.$$

# Discretization

- As in the elliptic case $\mathcal{C}$ is infinite, but we have exponential decay.
- This allows us to consider a truncated problem.
- In doing so we commit only an exponentially small error

$$I^{1-\gamma}\|tr_{\Omega}(\mathcal{U}-v)\|^2_{L^2(\Omega)} + \|\nabla(\mathcal{U}-v)\|^2_{L^2(0,T;L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}}))} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}}.$$

- For $\gamma = 1$[*], we consider backward Euler:
  - We initialize by setting $V^0(x,0) = \mathsf{u}_0$.
  - For $k = 0, \ldots, \mathcal{K}-1$, we find $V^{k+1} \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}})$ solution of

$$\tau^{-1}(V^{k+1}(\cdot,0) - V^k(\cdot,0), W(\cdot,0))_{L^2(\Omega)} + a(V^{k+1}, W) = (f^{k+1}, W(\cdot,0))_{L^2(\Omega)}$$

    for all $W \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}})$, where $f^{k+1} = f(t^{k+1})$.
  - Unconditional stability:

$$\|V^\tau(\cdot,0)\|^2_{\ell^\infty(L^2(\Omega))} + \|V^\tau\|^2_{\ell^2(\mathring{H}^1_L(y^\alpha,\mathcal{C}_{\mathcal{Y}}))} \lesssim \|\mathsf{u}_0\|^2_{L^2(\Omega)} + \|f^\tau\|^2_{\ell^2(\mathbb{H}^{-s}(\Omega))}.$$

[*] For $\gamma < 1$ see Nochetto, Otárola, AJS 2016

# Error estimates for fully discrete schemes

Discretization in time and space: stability + consistency yield

- Error estimates for $\mathcal{U}$: $s \in (0,1)$ and $\gamma \in (0,1)$

$$[I^{1-\gamma}\|tr_\Omega(v^\tau - V^\tau_{\mathscr{T}_\mathcal{Y}})\|^2_{L^2(\Omega)}(T)]^{\frac{1}{2}} \lesssim \tau^\theta + |\log \# \mathscr{T}_\mathcal{Y}|^{2s} \# \mathscr{T}_\mathcal{Y}^{\frac{-(1+s)}{n+1}}$$

$$\|v^\tau - V^\tau_{\mathscr{T}_\mathcal{Y}}\|_{\ell^2(\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y}))} \lesssim \tau^\theta + |\log \# \mathscr{T}_\mathcal{Y}|^{s} \# \mathscr{T}_\mathcal{Y}^{\frac{-1}{n+1}}.$$

- Error estimates for $u$: $s \in (0,1)$ and $\gamma \in (0,1)$

$$[I^{1-\gamma}\|u^\tau - U^\tau\|^2_{L^2(\Omega)}(T)]^{\frac{1}{2}} \lesssim \tau^\theta + |\log \# \mathscr{T}_\mathcal{Y}|^{2s} \# \mathscr{T}_\mathcal{Y}^{\frac{-(1+s)}{n+1}}$$

$$\|u^\tau - U^\tau\|_{\ell^2(\mathbb{H}^s(\Omega))} \lesssim \tau^\theta + |\log \# \mathscr{T}_\mathcal{Y}|^{s} \# \mathscr{T}_\mathcal{Y}^{\frac{-1}{n+1}},$$

where $\theta < \frac{1}{2}$.

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Nochetto, Otárola, AJS 2016

# Outline

# Formulation

- Given $f \in \mathbb{H}^{-s}(\Omega)$ and an obstacle $\psi \in \mathbb{H}^s(\Omega) \cap C(\bar{\Omega})$ with $\psi \leq 0$ on $\partial\Omega$.
- Find $u \in \mathcal{K}$ such that

$$\langle (-\Delta)^s u, u - w \rangle \leq \langle f, u - w \rangle \quad \forall w \in \mathcal{K}$$

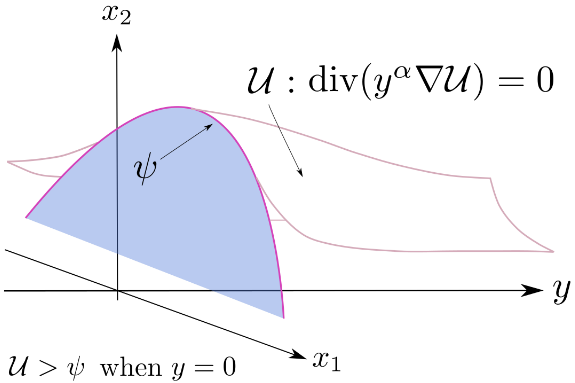
  where

$$\mathcal{K} := \{ w \in \mathbb{H}^s(\Omega) : \; w \geq \psi \text{ a.e. in } \Omega \}.$$

- Nonlinear and (because of $(-\Delta)^s$) nonlocal problem!
- Use the Caffarelli-Silvestre extension.

# Thin obstacle problem

- We convert the fractional obstacle problem into a thin obstacle problem.




- The restriction $\mathcal{U} > \psi$ only applies when $y = 0$ (thin obstacle).

# Truncation

- The domain $\mathcal{C}$ is infinite.
- The energy of the solution decays exponentially in $y$.
- We truncate the cylinder $\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y})$ and consider a truncated problem.
- In doing this we only commit an exponentially small error

$$\|\nabla(\mathcal{U} - \mathcal{V})\|_{L^2(y^\alpha, \mathcal{C}_{\mathcal{Y}})} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/8}.$$

# Discretization

Discretize the truncation over an anisotropic mesh.

## Theorem ([1])

*If $\mathcal{U}$ is the exact solution and $V_{\mathscr{T}_\mathcal{Y}}$ the discrete solution, then*

$$\|\mathcal{U} - V_{\mathscr{T}_\mathcal{Y}}\|_{\mathring{H}^1_L(y^\alpha,\mathcal{C})} \lesssim |\log(\#\mathscr{T}_\mathcal{Y})|^s (\#\mathscr{T}_\mathcal{Y})^{-1/(n+1)},$$

*where $C$ depends on the Hölder moduli of smoothness of $\mathcal{U}$ and $\mathcal{V}$, $\|f\|_{\mathbb{H}^{-s}(\Omega)}$ and $\|\psi\|_{\mathbb{H}^s(\Omega)}$.*

- Optimal regularity in $\Omega$[2]: $u \in C^{1,s}$.
- This implies that $\partial^\alpha_\nu \mathcal{U}(\cdot, 0) \in C^{0,1-s}$.
- For $y$ *"small"* use that[3]: $s \leq \frac{1}{2} \Rightarrow \mathcal{V} \in C^{0,2s}(\mathcal{C}_\mathcal{Y})$ and $s > \frac{1}{2} \Rightarrow \mathcal{V} \in C^{1,2s-1}(\mathcal{C}_\mathcal{Y})$.
- For $y$ *"big"* use[4] $\mathcal{V} \in H^2(y^\beta, \mathcal{C}_\mathcal{Y})$ with $\beta > 1 + 2\alpha$.

Back

[1] Nochetto, Otárola, AJS 2015

[2] Caffarelli, Salsa and Silvestre 2008

[3] Allen, Lindgren, and Petrosyan 2014

[4] Nochetto, Otárola, AJS 2015

# Outline

# Formulation

- Define the energy

$$\mathcal{J}(v) = \frac{1}{2}\|v\|^2_{\mathbb{H}^s(\Omega)} + \mathbf{1}_{\mathcal{K}}(v).$$

- We will study the (sub)gradient flow

$$u_t + \partial\mathcal{J}(u) \ni f \qquad u|_{t=0} = u_0.$$

- Equivalently we have the evolution variational inequality

$$(u_t, u - \phi)_{L^2(\Omega)} + \langle(-\Delta)^s u, u - \phi\rangle \leq (f, u - \phi)_{L^2(\Omega)} \quad \forall \phi \in \mathcal{K}.$$

- Or the complementarity conditions

$$\min\{u_t + (-\Delta)^s u - f, u - \psi\} = 0.$$

# The Caffarelli-Silvestre extension and truncation

- We will again overcome the nonlocality with the Caffarelli-Silvestre extension and consider

$$(\mathcal{U}_t(\cdot,0),(\mathcal{U}-\phi)(\cdot,0))_{L^2(\Omega)} + \frac{1}{d_s}\int_{\mathcal{C}} y^\alpha \nabla\mathcal{U}\nabla(\mathcal{U}-\phi)\,\mathrm{d}x\,\mathrm{d}y \leq (f,(\mathcal{U}-\phi)(\cdot,0))_{L^2(\Omega)}$$

  for all $\phi \in \mathring{H}^1_L(y^\alpha,\mathcal{C})$ with $\phi(\cdot,0)\in\mathcal{K}$.

- We consider, again, a truncated problem over $\mathcal{C}_\Upsilon$:

$$\|(\mathcal{U}-\mathcal{V})(\cdot,0)\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathcal{U}-\mathcal{V}\|_{L^2(0,T;\mathring{H}^1_L(y^\alpha,\mathcal{C}_\Upsilon))} \lesssim e^{-\sqrt{\lambda_1}\Upsilon/8}$$

# Time discretization

- The energy $\mathcal{J}$ is convex and lower semicontinuous $\Longrightarrow \partial\mathcal{J}$ is maximal monotone.
- We use the implicit Euler method:

$$\left(\frac{V^{k+1}-V^k}{\tau}(\cdot,0),(V^{k+1}-\phi)(\cdot,0)\right)_{L^2(\Omega)}$$

$$+\frac{1}{d_s}\int_{\mathcal{C}_\Upsilon} y^\alpha \nabla V^{k+1}\nabla(V^{k+1}-\phi)\,\mathrm{d}x\,\mathrm{d}y \leq \left(f^{k+1},(V^{k+1}-\phi)(\cdot,0)\right)_{L^2(\Omega)}$$

for all $\phi \in \mathring{H}^1_L(y^\alpha,\mathcal{C})$ with $\phi(\cdot,0)\in\mathcal{K}$.

# Time discretization

The general theory of graident flows[📘] yields:

- If $u_0 \in \mathcal{K}$ and $f \in L^2(0,T;L^2(\Omega))$

$$\|(\mathcal{V}-V)(\cdot,0)\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathcal{V}-V\|_{L^2(0,T;\mathring{H}_L^1(y^\alpha,\mathcal{C}_\mathcal{Y}))} \lesssim \tau^{1/2}.$$

- If $u_0 \in \mathcal{K} \cap \mathbb{H}^{2s}(\Omega)$ and $f \in BV(0,T;L^2(\Omega))$

$$\|(\mathcal{V}-V)(\cdot,0)\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathcal{V}-V\|_{L^2(0,T;\mathring{H}_L^1(y^\alpha,\mathcal{C}_\mathcal{Y}))} \lesssim \tau.$$

These estimates are sharp!

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[📘] Nochetto, Savaré, Verdi 2000

# Space discretization I: Minimal regularity

- Discretize in space using finite elements over an anisotropic mesh $\mathscr{T}_{\mathcal{Y}}$.
- If the discrete initial condition $V^0_{\mathscr{T}_{\mathcal{Y}}}$ satisfies

$$\|\nabla V^0_{\mathscr{T}_{\mathcal{Y}}}\|_{L^2(y^\alpha,\mathcal{C}_{\mathcal{Y}})} \lesssim \|u_0\|_{\mathbb{H}^s(\Omega)}.$$

then

$$\|(V - V_{\mathscr{T}_{\mathcal{Y}}})(\cdot,0)\|_{L^\infty(0,T;L^2(\Omega))} + \|V - V_{\mathscr{T}_{\mathcal{Y}}}\|_{L^2(0,T;\mathring{H}^1_L(y^\alpha,\mathcal{C}_{\mathcal{Y}}))} \lesssim \tau^\theta + \|\mathcal{V} - \Pi\mathcal{V}\|^{1/2}_{L^2(0,T;\mathring{H}^1_L(y^\alpha,\mathcal{C}_{\mathcal{Y}}))}.$$

where $\theta \in \{1/2, 1\}$ depends on the smoothness of $f$ and $u_0$

- No regularity assumptions!

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Otárola, AJS 2016

# Space discretization II: Analysis with regularity

- Under certain conditions we have that[1]

$$u_t, (-\Delta)^s u \in \mathsf{logLip}((0,T], C^{1-s}(\bar{\Omega})) \quad s \leq \frac{1}{3},$$

$$u_t, (-\Delta)^s u \in C^{\frac{1-s}{2s}}((0,T], C^{1-s}(\bar{\Omega})) \quad s > \frac{1}{3}.$$

- With this regularity[2]

$$\|(V - V_{\mathscr{T}_\mathcal{Y}})(\cdot,0)\|_{L^\infty(0,T;L^2(\Omega))} + \|V - V_{\mathscr{T}_\mathcal{Y}}\|_{L^2(0,T;\mathring{H}^1_L(y^\alpha,\mathcal{C}_\mathcal{Y}))} \lesssim$$
$$\tau + |\log \#\mathscr{T}_\mathcal{Y}|^s \left( \#\mathscr{T}_\mathcal{Y}^{-\frac{1}{n+1}} + \frac{\#\mathscr{T}_\mathcal{Y}^{-\frac{1+s}{n+1}}}{\tau^{1/2}} \right)$$
$$+ \|\mathcal{V} - \Pi\mathcal{V}\|_{L^2(0,T;\mathring{H}^1_L(y^\alpha,\mathcal{C}_\mathcal{Y}))}$$

Back

[1] Caffarelli and Figalli 2013

[2] Otárola, AJS 2016