Linear Algebra Self-Test

In this work sheet, you can examine your own competence in Linear Algebra. Linear Algebra is a key mathematical tool in many engineering disciplines, particularly in AI, Computer Vision, and Edge Computing.

The aim is to see if you can answer all (or at least most) of the questions below.

If you cannot, you must be in a position to prepare yourself before commencing the programme. We recommend that you undertake the following free course:

https://www.khanacademy.org/math/linear-algebra

Vector operations:

1. Let $u = \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}$ and $v = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Find:

(a) $3u$ (answer: $[9, \quad -12, \quad 21]^\top$)

(b) $u + v$ (answer: $[1, \quad -4, \quad 8]^\top$)

(c) $3u - 4v$ (answer: $[17, \quad -12, \quad 17]^\top$)

2. Find x and y where:

(a) $\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$ (answer: $x = 2$ and $y = 3$)

(b) $\begin{bmatrix} x \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ x + y \end{bmatrix}$ (answer: $x = 2$ and $y = 1$)

(b) $\begin{bmatrix} 4 \\ y \end{bmatrix} = x \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (answer: $x = 2$ and $y = 6$)

3. Find $u \cdot v$, where $\cdot$ represents the dot product:

(a) $u = [3, -4, 7]^\top$ and $v = [-2, 0, 1]^\top$ (answer: 1)

(b) $u = [4, 2, -3, 5, -1]^\top$ and $v = [2, 6, -1, -4, 8]^\top$ (answer: -5)

4. Consider the following pairs of vectors. Are they orthogonal (perpendicular)? [hint: we're still thinking dot product]

(a) $u = [3, -4, 7]^\top$ and $v = [-2, 0, 1]^\top$ (no answers, these are yes/no)

(b) $u = [4, 2, -3, 5, -1]^\top$ and $v = [2, 6, -1, -4, 8]^\top$

(c) $u = [5, 4, 1]^\top$ and $v = [1, -2, 3]^\top$

Matrix operations:

1. Given $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 2 \\ -7 & 1 & 8 \end{bmatrix}$, calculate:

(a) $A + B$ (answer: $\begin{bmatrix} 4 & -2 & 5 \\ -3 & 6 & 2 \end{bmatrix}$)

(b) $3A$ (answer: $\begin{bmatrix} 3 & -6 & 9 \\ 12 & 15 & -18 \end{bmatrix}$)

(c) $2A - 3B$ (answer: $\begin{bmatrix} -7 & -4 & 0 \\ 29 & 7 & -36 \end{bmatrix}$)

2. Find the transpose of the following matrices:

(a) $\begin{bmatrix} 1 \\ -4 \\ 8 \end{bmatrix}$ (no answers, these are easy)

(b) $\begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 \\ 7 & 8 & -9 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

(e) Why is the matrix in (d) called symmetric?

3. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$, find:

(a) AB (ans: $AB = \begin{bmatrix} 11 & -6 & 14 \\ 1 & 2 & -14 \end{bmatrix}$)

(b) BA

4. We let $(r \times c)$ denote a matrix with r rows and c columns. Are the following matrix products defined, and if so, what size is the resulting matrix?

(a) $(2 \times 3)(3 \times 4)$ (b) $(1 \times 2)(3 \times 1)$ (c) $(4 \times 4)(3 \times 3)$

(d) $(4 \times 1)(1 \times 2)$ (e) $(5 \times 2)(2 \times 3)$ (f) $(2 \times 2)(2 \times 4)$

5. Find the matrix-vector product:

(a) $\begin{bmatrix} 1 & 6 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -7 \end{bmatrix}$ (ans: $\begin{bmatrix} -40 \\ -41 \end{bmatrix}$)

(b) $\begin{bmatrix} 1 & 6 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -7 \end{bmatrix}$ (ans: $\begin{bmatrix} -40 \\ -41 \\ -7 \end{bmatrix}$)

(c) $\begin{bmatrix} 1 & 6 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -7 \\ 1 \end{bmatrix}$ (ans: …)

6. Given $A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$, find a non-zero column vector $u = \begin{bmatrix} x \\ y \end{bmatrix}$ such that $Au = 3u$. (ans: $u = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$)

Determinant and Inverses

6. Given $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, sketch on a graph the geometric representation of the determinant

7. Which of the following are invertible? Find the inverse, where appropriate:

(a) $A = \begin{bmatrix} 5 & 2 \\ 4 & 1 \end{bmatrix}$ (ans: $A^{-1} = \begin{bmatrix} -1/3 & 2/3 \\ 4/3 & -5/3 \end{bmatrix}$)

(b) $B = \begin{bmatrix} 2 & -3 \\ 1 & 3 \end{bmatrix}$

(c) $C = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix}$

8. Show that $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$ are inverses. Hint, $AA^{-1} = ?$.

Eigenvalues and Eigenvectors

9. Find the eigenvalues and eigenvectors of the following matrices. Can the matrices be diagonalised? If the matrix A can be diagonalized, find the diagonalised matrix D (i.e., $P^{-1}AP = D$).

(a) $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ (ans: $\lambda = -1$ and $\lambda = 5$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$)

(b) $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (ans: $\lambda = -1$, $\lambda = 1$, and $\lambda = 2$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$)

(c) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (ans: $\lambda = 1$, and $\lambda = 2$, $D = ?$)

10. Given the matrix A below, use the spectral theorem to find A^{50}.

(a) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ (ans: $A^{50} = \begin{bmatrix} 2^{50} & 2^{50} - 1 & 0 \\ 0 & 1 & 0 \\ 1 - 2^{50} & 1 - 2^{50} & 1 \end{bmatrix}$)